Fermi Theory of Beta Decay (Contd.)

Course: MPHYCC-13 Nuclear and Particle Physics
(M.Sc. Sem-III)

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Fermi theory of beta decay (previous class)

$|\psi_i>$ → $|\psi_f>$

Initial state represents the state vector/ wave function of parent nucleus

Final state represents the combined state vector/wave function of duaghter nucleus and decay particles (beta particles and neutrinos)

The transition probability (rate) for the decay is:

$$\lambda = \frac{2\pi}{\hbar} |H_{if}^{p}|^2 \frac{dn}{dE_f}$$

where the matrix elements $H_{if}^{p} = \int \psi_f^* H^p \psi_i d\tau$ with H^p representing the interaction potential responsible for beta decay.

The matrix elements modify as:

$$H_{if}^{p} = \int \psi_d^* \psi_e^* \psi_\nu^* H^p \psi_p d\tau$$

The wave functions of beta particle and neutrino have the usual free particle's wave function form normalized within the volume V (which is nuclear volume for beta decay case).

$$\psi_e = \frac{1}{\sqrt{V}} e^{i\frac{\boldsymbol{p}_e \cdot \boldsymbol{r}}{\hbar}}$$

$$\psi_\nu = \frac{1}{\sqrt{V}} e^{i\frac{\boldsymbol{p}_\nu \cdot \boldsymbol{r}}{\hbar}}$$

Under the approximation of $\frac{p\,r}{\hbar} \ll 1$,

$$\psi_e \approx \frac{1}{\sqrt{V}}$$

$$\psi_\nu \approx \frac{1}{\sqrt{V}}$$

This approximation is known as the **allowed approximation**.

Now, the matrix element:

$$H_{if}^{p} = \frac{1}{V} \int \psi_d{}^{*} H^{p} \psi_p \, d\tau$$

$$H_{if}^{p} = \frac{1}{V} M_{if} \qquad \text{---------- (10)}$$

where $M_{if} = \int \psi_d{}^{*} H^{p} \psi_p \, d\tau$ is known as **nuclear matrix elements** as only the waves of parent and daughter nucleus involve in the expression.

Now, the updated the expression of transition rate

$$\lambda = \frac{2\pi}{\hbar} \frac{1}{V^2} |M_{if}|^2 \frac{dn}{dE_f}$$

Then, the total number of final states which have simultaneously an electron and a neutrino (confined in spatial volume V) with momenta p to p+dp and q to q+dp are:

$$dn = \frac{(4\pi)^2 V^2 p^2 dp\, q^2 dq}{h^6}$$

$$\lambda = \frac{2\pi}{\hbar}|M_{if}|^2(4\pi)^2\frac{p^2\,dp\,q^2}{h^6}\frac{dq}{dE_f} \quad \text{----------- (1)}$$

This much we discussed in the previous class.

Nuclear Matrix elements (M_{if}) can treated as constant becuase, we can consider the nuclear potentials for the parent and duaghter nuclei are time independent.

Now, we notice that, energy of final quantum states (E_f) can be given as:

$$E_f = E_e + E_v \qquad \text{--------- (2)}$$

where E_e is total reletivistic energy of electron while E_v is for neutrino.

$$E_e = m_e c^2 + K \quad \text{(K is kinetic energy of electrons) ----(3)}$$

and $E_v^2 = q^2 c^2 + m_v^2 c^4$ (q is momentum of neutrino)

As neutrino mass can approximated to zero,

$$E_v^2 = q^2 c^2$$

$$E_v = q\,c \qquad \text{-------- (4)}$$

Using equations (3) and (4) into (2), we get

$$E_f = m_e c^2 + K + q\,c$$

$$dE_f = c\,dq$$

$$\frac{dq}{dE_f} = \frac{1}{c} \qquad \text{--------- (5)}$$

Use this result in equation (1), we get

$$\lambda = \frac{2\pi}{\hbar} |M_{if}|^2 (4\pi)^2 \frac{p^2 dp\, q^2}{h^6} \frac{dq}{dE_f}$$

$$\lambda = \frac{2\pi}{\hbar} |M_{if}|^2 (4\pi)^2 \frac{p^2 dp\, q^2}{h^6} \frac{1}{c}$$

$$\lambda = \frac{2\pi}{\hbar} \frac{|M_{if}|^2 (4\pi)^2}{h^6 c} q^2 p^2 dp$$

$$\lambda = C_0 q^2 p^2 dp \qquad \text{----------- (6)}$$

where $C_0 = \frac{2\pi}{\hbar} \frac{|M_{if}|^2 (4\pi)^2}{h^6 c}$ is assumed to be constant as M_{if} is taken to be constant.

Now note that λ in equation (6) represents the probibility (per unit time) of the system to make transition from initial state to final state. In other words, it tells us the probibility of an nucleus to undergo beta decay per unit time.

In the same sense, we can say that λ provides us the number of beta particles having momentum *p* to *p+dp* and given by N(p) dp:

$$N(p)\, dp = C_0 q^2 p^2 dp \qquad \text{---------- (7)}$$

To relate this above theoritical expression with exprimental results, we need to write it in the form of kinetic energy of beta particles. For that,

$$E_e = \sqrt{p^2 c^2 + m_e^2 c^4} = K + m_e c^2 \quad \text{--------(8)}$$

$$p^2 c^2 + m_e^2 c^4 = (K + m_e c^2)^2$$

$$p^2 c^2 = K(K + 2 m_e c^2)$$

$$p^2 = \frac{K}{c^2}(K + 2 m_e c^2) \quad \text{------------} \quad (9)$$

Moreover, from the previous lectures, we know that Q-value in beta decay is shared in the form of kinetic energy of beta particle and energy of neutrino. Therefore,

$$Q = K + qc$$

$$q = \frac{(Q - K)}{c}$$

$$q^2 = \frac{(Q - K)^2}{c^2} \quad \text{-----------} \ (10)$$

From equation (9)

$$p = \frac{\sqrt{K(K + 2 m_e c^2)}}{c}$$

$$dp = \frac{1}{c} \frac{\left(2K + 2m_e c^2\right)}{2\sqrt{K\left(K + 2m_e c^2\right)}} dK \qquad \text{------ (11)}$$

Use eqautions (10) into equation (7), we get

$$N(p)\,dp = C_0 q^2 p^2 dp$$

$$N(p)\,dp = C_0 \frac{(Q-K)^2}{c^2} p^2 dp \qquad \text{------- (11 a)}$$

Using expression of K written in equation (8) in the above equation, we get

$$K = \sqrt{p^2 c^2 + m_e{}^2 c^4} - m_e c^2$$

$$N(p)\,dp = \frac{C_0}{c^2}\left(Q + m_e c^2 - \sqrt{p^2 c^2 + m_e{}^2 c^4}\right)^2 p^2 dp$$

$$N(p) = \frac{C_0}{c^2}\left(Q + m_e c^2 - \sqrt{p^2 c^2 + m_e{}^2 c^4}\right)^2 p^2 \qquad \text{------ (12)}$$

N(p) provides the number of beta particles emitted with momentum p. This is the distributation of beta particles in terms of their momentum. Next, we would like to write the distributation in terms of their kinetic energy.

For the distributation in terms of energy, we use equation (9), (10) and (11) in equation (7) {i.e., we replace both the momenta p and q in terms of K}

$$N(p)\,dp = C_0 q^2 p^2 dp$$

As momenta q and p are replaced by energy, N(p)dp also gets replaced by N(K)dK which represents of number of beta particles having kinetic energy K to K+dK.

$$N(K)\,dK = C_0 \frac{(Q-K)^2}{c^2} \frac{K(K+2m_e c^2)}{c^2} \frac{1}{c} \frac{(2K+2m_e c^2)}{2\sqrt{K(K+2m_e c^2)}} dK$$

$$N(K)\,dK = \frac{2C_0}{2c^5}(Q-K)^2\sqrt{K(K+2m_e c^2)}(K+m_e c^2)\,dK$$

$$N(K)\,dK = C_1(Q-K)^2\sqrt{K(K+2m_e c^2)}(K+m_e c^2)\,dK$$

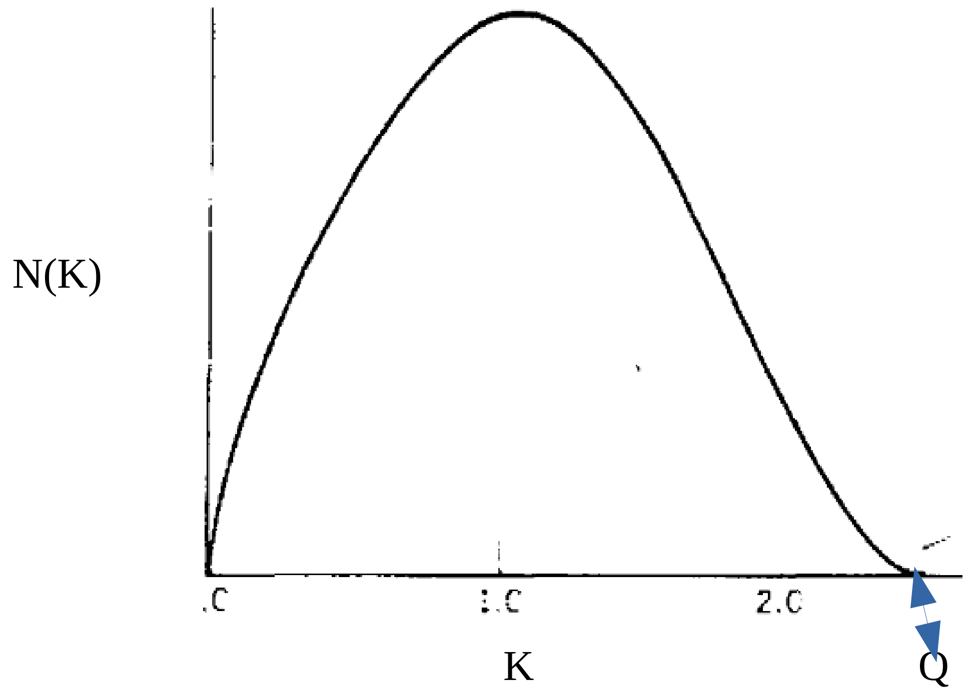
$$N(K) = C_1(Q-K)^2\sqrt{K(K+2m_e c^2)}(K+m_e c^2) \quad \text{----(11)}$$

This distributation is called the Fermi-expression in terms of the kinetic energy. The distributation provides the number of beta particles those emitted with kinetic energy K.

$$N(K) = C_1(Q-K)^2\sqrt{K(K+2m_e c^2)}(K+m_e c^2)$$

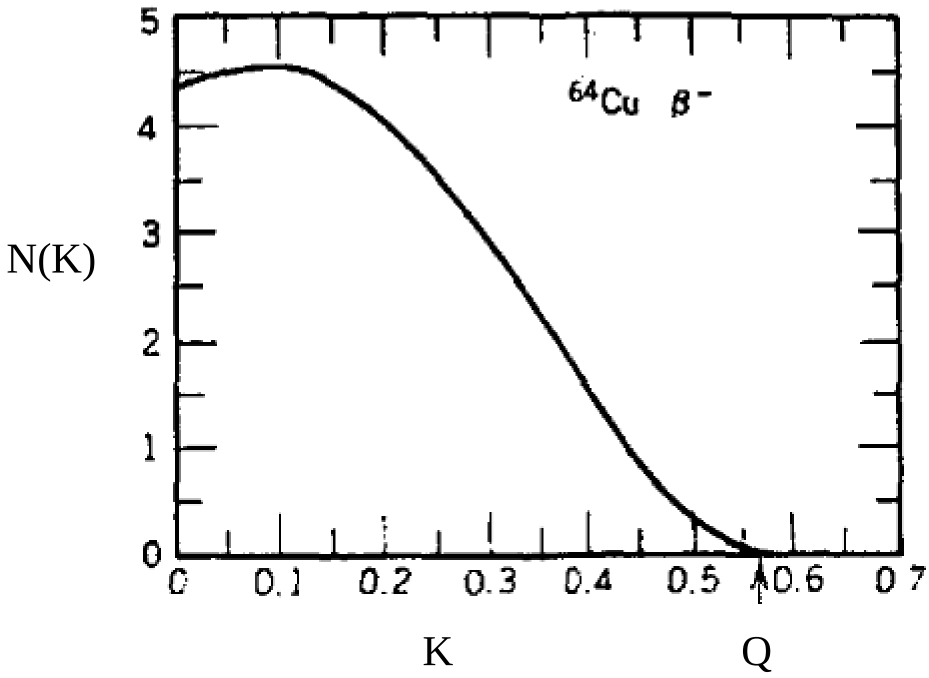
Note that for K=0 and K=Q, N(K)=0 => It means that number of beta particles having kinetic energy 0 and Q is zero. And if we plot the N(K) vs. K, we obtain a continuous energy distributation.

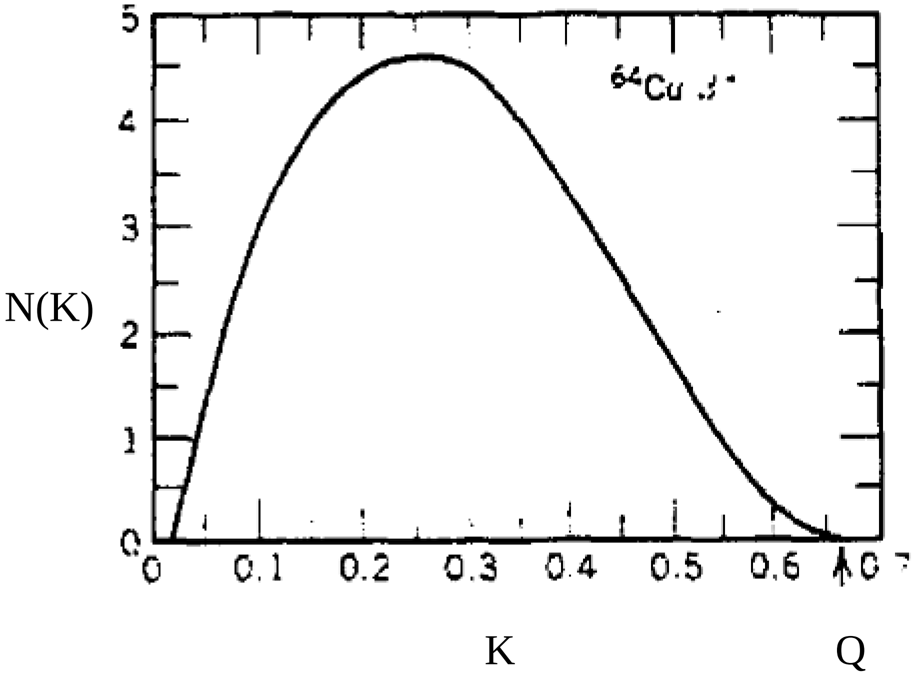
If plot N(K) vs. K for Q=2.5 MeV, we obtain following curve:



(Plot from Fermi theory)

Now, if we see the experimentally obtained plots:





There are differences between the plot obtained from theory and experimental plot. The different is more evident in the case of β^- decay.
These differences originate because we didn't consider the Coulomb interaction between the beta particle and the daughter nucleus.

Classically. we can interpret the shapes of the experimentally obtained energy distributions as a Coulomb repulsion of β^+ by the nucleus, giving fewer low-energy positrons, and a Coulomb attraction of β^-, giving more low-energy electrons.

From the more accurate calculations, we should use quantum mechanics to study the change in the electron/positron plane wave under the nuclear Coloumb potential. It modifies the energy spectrum/distributation by introducing an additional factor, the Fermi function $F(Z', K)$ where Z' is atomic number of daughter nucleus.

$$N(K)=C_1 F(Z', K)(Q-K)^2\sqrt{K(K+2m_e c^2)}(K+m_e c^2)$$

This explains the exprimentally obtained energy spectrum of beta spectrum.

Furthermore, in some cases, we need to include the effect of the nuclear matrix element M_{if}, which we have up to now constant and assumed not to influence the shape of the spectrum. This approximation (also called the allowed approximation) is often found to be a very good one. but there are some decays in which it doesn't work. Such decays are known as forbidden dicay.

So overall, we have

$$N(K)\propto F(Z', K)(Q-K)^2\sqrt{K(K+2m_e c^2)}(K+m_e c^2)|M_{if}|^2$$

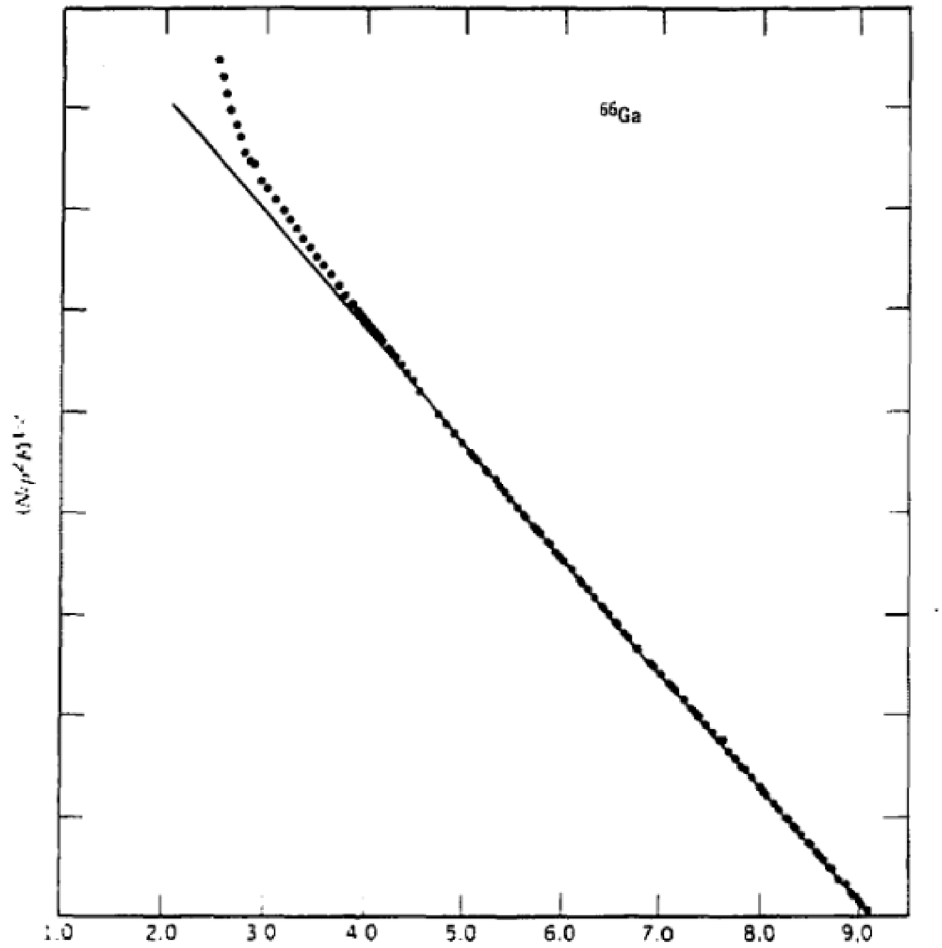
Just for the book-keeping, if we utilize the momentum distributation N(p) written in equation 11(a), we can get

$$N(p)\propto p^2(Q-K)^2 F(Z', p)|M_{if}|^2$$

Under allowed approximation, M_{if} can also be taken constant then the above relation can be written as:

$$(Q-K)\propto\sqrt{\frac{N(p)}{p^2 F(Z', p)}}$$

Note that Plot $\sqrt{\frac{N(p)}{p^2 F(Z', p)}}$ and K will be a straight line which intercepts the x-axis at the decay energy Q. Such a plot is called a Kurie plot (sometimes a Fermi plot or a Fermi-Kurie plot).



Thanks for the attention!