

Fermi Theory of Beta Decay



**Course: MPHYCC-13 Nuclear and Particle Physics
(M.Sc. Sem-III)**

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Key results of Time-dependent perturbation theory

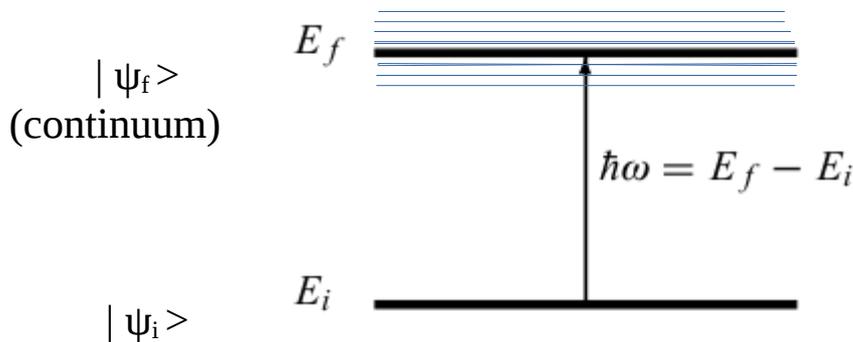
In time-dependent perturbation theory, we split the Hamiltonian of a quantum system into two parts. First part is time independent H_0 and other is time dependent $V(t)$ such that $V(t)$ is small compared to H_0

$$H = H_0 + V(t)$$

Because of the time dependent perturbing hamiltonian, the system can make transition from its initial state to some final state.

Example: An atom is placed under the influence of an electromagnetic wave (oscillating electric field). Electrons experience oscillatory electric force and therefore the Hamiltonian becomes time-dependent. This may lead to transition of atom from ground state to possible excited states.

First order transition rate (W_{if}) for the transition from the initial state $|\psi_i\rangle$ to continuum final state $|\psi_f\rangle$ is written as follows.



$$W_{if} = \frac{2\pi}{\hbar} \left| \langle \psi_f | \hat{V} | \psi_i \rangle \right|^2 \rho(E_f)$$

Fermi's Golden rule

where in discrete basis, $V_{if} = \langle \psi_f | \hat{V} | \psi_i \rangle$ represent the matrix elements of the perturbing hamiltonian. In position (continuous) basis, the matrix elements are given as $\langle \psi_f | \hat{V} | \psi_i \rangle = \int \psi_f^* V \psi_i d\tau$

Note that transition rate represents the transition probability per unit time.

Here $\rho(E_f)$ represents the density of final states. It is the number of states per unit energy interval at E_f .

Few points to keep in mind:

1. Density of states must be included for the following reason: if the final state E_f is a single isolated state, then the decay probability will be much smaller than it would be in the case that there are many states in a narrow band near E_f . If there is a large density of states near E_f , there are more possible final states that can be reached by the transition and thus a larger transition probability.
 2. In beta decay process, the beta particle and neutrino behave as free particles and therefore lead to large number of final quantum states.
 3. Even though a system may make a transition from an initial energy state E_i to a final state E_f , energy must be conserved. Thus the total decay energy must be constant. If the final state E_f is of lower energy than E_i , the energy difference $E_i - E_f$ must appear as radiation emitted in the decay. In transitions between atomic or nuclear excited states, a photon is emitted to carry the energy $E_i - E_f$.
 4. **The transition rate is directly associated with the decay constant λ which also represent the probability of decaying any atom at a given instant of time. And from λ , one can obtain the average life and half-life.**
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Fermi theory of beta decay

In 1934, Fermi formulated a successful theory of beta decay which is based on Pauli's neutrino hypothesis. Fermi theory provides an expression for the transition probability (or rate) for beta decay.

The theory is based on following considerations:

1. The electron and neutrino do not exist before the decay process, and therefore the theory must account for the formation of those particles.
2. The electron and neutrino must be treated relativistically.
3. The continuous distribution of electron energies must result from the calculation.
4. **An interaction causing beta decay is weak compared with the interaction that forms the quasi-stationary states. In other words, the (time-dependent) potential responsible for beta decay is small compared to the nuclear potential (time independent) which forms stationary states. Therefore, time-dependent perturbation theory can be applied to the beta decay process, as we can treat decay-causing interaction as weak perturbation.**

Lets consider the total Hamiltonian for the system is

$$H = H_N + H^P(t)$$

where $H^P(t)$ is considered to be responsible for beta decay and is smaller than nuclear potential H_N .

Now, we know in beta decay:



Initial state represents the state vector/ wave function of parent nucleus

Final state represents the combined state vector/wave function of daughter nucleus and decay particles (beta particles and neutrinos)

Then, from the Fermi Golden rule, the transition rate from initial state to final state or the decay probability can be given as:

$$\lambda = \frac{2\pi}{\hbar} |\langle \psi_f | \hat{H}^p | \psi_i \rangle|^2 \rho(E_f)$$

$$\lambda = \frac{2\pi}{\hbar} |H_{if}^p|^2 \rho(E_f) \quad \text{----- (1)}$$

where the matrix elements $H_{if}^p = \int \psi_f^* H^p \psi_i d\tau$ with H^p representing the interaction potential responsible for beta decay. Also, note that here we use λ instead of W_{if} as both are same.

While $\rho(E_f)$ is the density of final states, which can also be written as

$$\rho(E_f) = \frac{dn}{dE_f} \quad \text{----- (2)}$$

dn represents the number of final states in the energy interval dE_f . As mentioned above, a given transition is more likely to occur if there is a large number of accessible final states.

As mentioned above, for beta decay

$$\psi_i \equiv \psi_p \text{ (wave function of parent nucleus)}$$

$$\psi_f \equiv \psi_d \psi_e \psi_\nu \text{ (combined wave function of daughter nucleus, beta-particle and anti-neutrino/neutrino)}$$

where ψ_e and ψ_ν represent the wave functions of electron and neutrino. ψ_d is wave function of daughter nucleus.

Therefore, the matrix elements modify as:

$$H_{if}^p = \int \psi_d^* \psi_e^* \psi_\nu^* H^p \psi_i d\tau \quad \text{----- (3)}$$

Now, after their production, the beta particle and neutrino move as free particles. Therefore, the corresponding wave functions have the usual free particle's wave function form normalized within the volume V (which is nuclear volume for beta decay case).

$$\psi_e = \frac{1}{\sqrt{V}} e^{i k_e \cdot r}$$

Since wave vector $k_e = \frac{\mathbf{p}_e}{\hbar}$ with \mathbf{p}_e being the momentum of electrons and therefore the electron wave function becomes;

$$\psi_e = \frac{1}{\sqrt{V}} e^{i \frac{\mathbf{p}_e \cdot \mathbf{r}}{\hbar}} \quad \text{----- (4)}$$

Similarly the wave function for neutrino is given as:

$$\psi_\nu = \frac{1}{\sqrt{V}} e^{i \frac{\mathbf{p}_\nu \cdot \mathbf{r}}{\hbar}} \quad \text{----- (5)}$$

Typically, the kinetic energy of beta particle is 1 MeV. Then, the momentum is $p_e = 1.4 \text{ MeV}/c$.

Passing Remark: Calculation of momentum:

Relativistic energy $E = \sqrt{p^2 c^2 + m^2 c^4}$

Also, we know that the relativistic energy is sum of kinetic energy and rest mass energy i.e., $E = T + mc^2$

As a result, $T + mc^2 = \sqrt{p^2 c^2 + m^2 c^4}$

This implies:

$$(T + mc^2)^2 = p^2 c^2 + m^2 c^4$$

$$p^2 c^2 = (T + mc^2)^2 - m^2 c^4$$

$$p = \frac{\sqrt{(T + mc^2)^2 - m^2 c^4}}{c}$$

We are given the kinetic energy $T = 1 \text{ MeV}$ and we know the rest mass energy of electron (mc^2) = 0.511 MeV. Using these number in the above formula, we can get $p = 1.4 \text{ MeV} / c$.

With the momentum $p_e = 1.4 \text{ MeV}/c$, if we calculate

$$\frac{p_e}{\hbar} = \frac{1.4 \times 1.6 \times 10^{-13} \text{ J}}{3 \times 10^8 \text{ m/s}} \frac{1}{1.054 \times 10^{-34} \text{ J-s}}$$

$$\frac{p_e}{\hbar} = \frac{0.708}{10^{-13} \text{ m}}$$

$$\frac{p_e}{\hbar} \approx \frac{0.007}{10^{-15} \text{ m}} = 0.007 \text{ fm}^{-1}$$

For a typical nuclear radius $r = 1 \text{ fm}$,

$$\frac{p_e r}{\hbar} \approx 0.007$$

Therefore,

$$\frac{p_e r}{\hbar} \ll 1 \quad \text{----- (6)}$$

Now, from equation (4), the electron wave function can be expanded as:

$$\psi_e = \frac{1}{\sqrt{V}} e^{i \frac{\mathbf{p}_e \cdot \mathbf{r}}{\hbar}}$$

$$\psi_e = \frac{1}{\sqrt{V}} \left(1 + i \frac{\mathbf{p}_e \cdot \mathbf{r}}{\hbar} + \dots \right) \quad \text{----- (7)}$$

Using the condition $\frac{p_e r}{\hbar} \ll 1$, we can keep only first term and all the higher order terms can be neglected. As a result,

$$\psi_e \approx \frac{1}{\sqrt{V}} \quad \text{----- (8)}$$

This approximation is known as the **allowed approximation**.

Under the allow approximation, we can also neglect the higher order term in the exponetial of the wave function of neutrino. As a result, the neutrino wavefunction written in equation (5) modifies as:

$$\psi_\nu \approx \frac{1}{\sqrt{V}} \quad \text{----- (9)}$$

Now, use the modified wavefunction of electron and neutrino written in equations (8) and (9) into equation (3) to calculate the matrix element:

$$H_{if}^p = \int \psi_d^* \psi_e^* \psi_\nu^* H^p \psi_i d\tau$$

$$H_{if}^p = \int \psi_d^* \frac{1}{\sqrt{V}} \frac{1}{\sqrt{V}} H^p \psi_i d\tau$$

$$H_{if}^p = \frac{1}{V} \int \psi_d^* H^p \psi_i d\tau$$

$$H_{if}^p = \frac{1}{V} M_{if} \quad \text{----- (10)}$$

where $M_{if} = \int \psi_d^* H^p \psi_i d\tau$ is known as **nuclear matrix elements** as only the waves of parent and daughter nucleus involve in the expression.

If we put the expression of H_{if}^p from equation (10) into equation (1), we get the updated the expression of transition rate

$$\lambda = \frac{2\pi}{\hbar} |H_{if}^p|^2 \rho(E_f)$$

$$\lambda = \frac{2\pi}{\hbar} \frac{1}{V^2} |M_{if}|^2 \frac{dn}{dE_f} \quad \text{----- (11)}$$

Now, to obtain the transition probability/rate, we need to calculate the density of states $\rho(E_f) = dn/dE_f$, while, at a time being, nuclear matrix element M_{if} can be treated as constant quantity. As a result, the transition probability basically governed by the density of final states. **Hence, the density of states determines (to lowest order) the shape of the beta energy spectrum.**

Note that daughter nucleus can be considered to have a final quantum state. In comparison, the decay products (electron and neutrino) being free particles can have continuum energy states. Therefore, to find the density

of states, we need to know the number of final states accessible to the decay products.

To obtain the number accessible quantum states for electron, let us suppose in the decay that we have an electron (or positron) emitted with momentum p_e . At present, the direction of momentum is not of our interest. One qualitative way to calculate the states for the momentum range p to $p+dp$ is as follows:

Since electron is considered to be free and, therefore, its position and momentum can be specified with uncertainties $dx, dy, dz, dp_x, dp_y, dp_z$ such that

$$\begin{aligned} dx dp_x &\sim h \\ dy dp_y &\sim h \\ dz dp_z &\sim h \end{aligned}$$

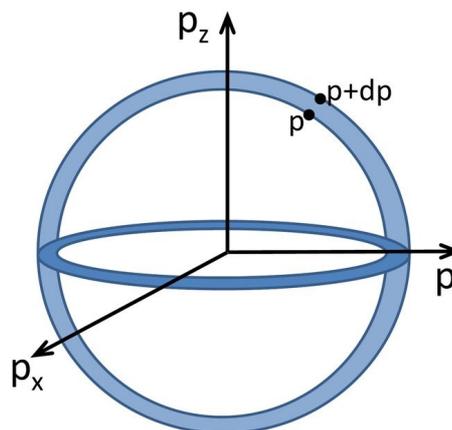
Then, we can say the smallest volume in phase which can be measured in quantum mechanics is:

$$dx dp_x dy dp_y dz dp_z \sim h^3$$

Or, in other words, we can also say that this volume corresponds the one quantum state of the particle as we can't determine the position and momentum inside the volume.

Now, if we want to calculate total number of quantum states inside a spatial volume V and momentum range p to $p+dp$, we need to calculate to the corresponding phase space volume by integration.

$$V_{phase} = \int dx dy dz \int_p^{p+dp} dp_x dp_y dp_z$$



$$V_{\text{phase}} = V 4 \pi p^2 dp$$

As one quantum state corresponds to h^3 phase volume in phase space, then number of quantum states corresponding to V_{phase} phase space volume are:

$$dn_e = \frac{4 \pi p^2 dp V}{h^3}$$

Basically, dn_e represents the number of quantum states available for electron confined in a spatial volume V and having momentum p to $p+dp$.

Similarly, we can calculate the number of quantum states available for neutrino confined in a spatial volume V and having momentum q to $q+dq$ and given as:

$$dn_\nu = \frac{4 \pi q^2 dq V}{h^3}$$

Then the total number of final states which have simultaneously an electron and a neutrino (confined in spatial volume V) with momenta p to $p+dp$ and q to $q+dp$ are:

$$dn = dn_e dn_\nu$$

$$dn = \frac{(4 \pi)^2 V^2 p^2 dp q^2 dq}{h^6} \quad \text{----- (12)}$$

If we use this expression of dn in equation (11), the expression for the transition probability modifies as:

$$\lambda = \frac{2 \pi}{\hbar} \frac{1}{V^2} |M_{if}|^2 \frac{dn}{dE_f}$$

$$\lambda = \frac{2 \pi}{\hbar} |M_{if}|^2 (4 \pi)^2 \frac{p^2 dp q^2}{h^6} \frac{dq}{dE_f}$$

Thanks for the attention!