Born Approximation

M.Sc. 4^{th} Semester

MPHYEC-1: Advanced Quantum Mechanics

Unit I

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Lecture Outline

- Green’s functions
- Scattering amplitude
- General solution of Schrödinger eq. in terms of Green’s function
- Born Series
- Assignment

Green's functions

The Green's function is obtained by solving the point source equation:

$$(\nabla^2 + k^2)G(\vec{r} - \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \qquad \ldots\ldots 1$$

where $G(\vec{r} - \vec{r}\,')$ and $\delta(\vec{r} - \vec{r}\,')$ are given by their Fourier transforms as follows:

$$\begin{aligned} G(\vec{r} - \vec{r}\,') &= \frac{1}{(2\pi)^3} \int e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')} \tilde{G}(\vec{q})\, d^3 q, \\ \delta(\vec{r} - \vec{r}\,') &= \frac{1}{(2\pi)^3} \int e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}\, d^3 q. \end{aligned} \qquad \ldots\ldots 2$$

Substituting Eqn (2) into Eqn(1) gives

$$\left(-\vec{q}^{\,2} + \vec{k}^{\,2}\right) \tilde{G}(\vec{q}) = 1 \qquad \Longrightarrow \qquad \tilde{G}(\vec{q}) = \frac{1}{\vec{k}^{\,2} - \vec{q}^{\,2}}. \qquad \ldots\ldots 3$$

The expression for $G(\vec{r}-\vec{r}\,')$ can be obtained by inserting (3) into (2)

$$G(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3}\int \frac{e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}}{k^2-q^2}d^3q. \qquad ......4$$

$$G(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3}\int_0^\infty \frac{q^2dq}{k^2-q^2}\int_0^\pi e^{iq|\vec{r}-\vec{r}\,'|\cos\theta}\sin\theta\,d\theta\int_0^{2\pi}d\varphi, \qquad ......5$$

To integrate over angle in (5) we need to make the variable change $x=cos\theta$

$$\int_0^\pi e^{iq|\vec{r}-\vec{r}\,'|\cos\theta}\sin\theta\,d\theta = \int_{-1}^1 e^{iq|\vec{r}-\vec{r}\,'|x}dx = \frac{1}{iq|\vec{r}-\vec{r}\,'|}\left(e^{iq|\vec{r}-\vec{r}\,'|}-e^{-iq|\vec{r}-\vec{r}\,'|}\right) \qquad ......6$$

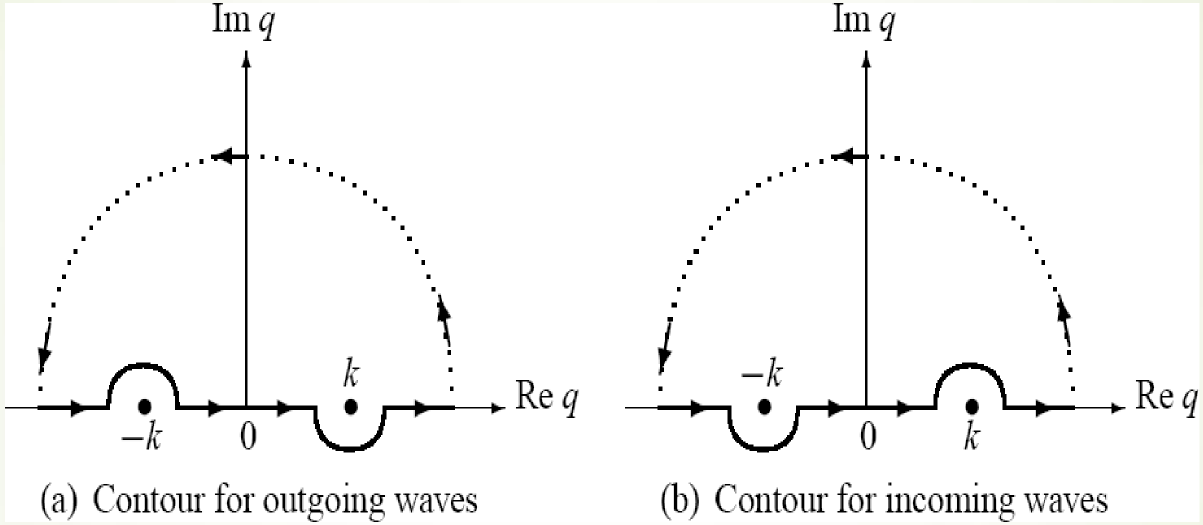
Thus, (4) becomes

$$G(\vec{r}-\vec{r}\,') = \frac{1}{4\pi^2 i|\vec{r}-\vec{r}\,'|}\int_0^\infty \frac{q}{k^2-q^2}\left(e^{iq|\vec{r}-\vec{r}\,'|}-e^{-iq|\vec{r}-\vec{r}\,'|}\right)dq,$$

or

$$G(\vec{r}-\vec{r}\,') = -\frac{1}{4\pi^2 i|\vec{r}-\vec{r}\,'|}\int_{-\infty}^{+\infty}\frac{qe^{iq|\vec{r}-\vec{r}\,'|}}{q^2-k^2}dq. \qquad ......7$$

The integral in (7) can be evaluated by the method of residues by closing the contour in the upper half of the q-plane:

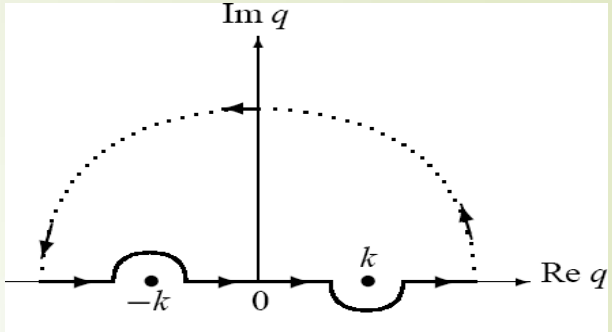


The integral is equal to $2\pi i$ times the residue of the integrand at the poles.

Since there are two poles, $q = \pm k$, the integral has two possible values:

- the value corresponding to the pole at $q = k$, which lies inside the contour of integration in Figure 1a, is given by

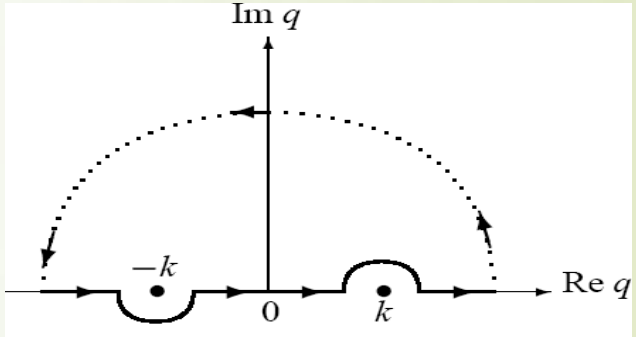
$$G_+(\vec{r} - \vec{r}\,') = -\frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \qquad \text{......8}$$



(a) Contour for outgoing waves

- the value corresponding to the pole at $q = -k$, Figure 1b, is

$$G_-(\vec{r} - \vec{r}\,') = -\frac{1}{4\pi}\frac{e^{-ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}. \qquad \text{......9}$$



(b) Contour for incoming waves

Green's function $G_+(\vec{r} - \vec{r}\,')$ represents an *outgoing spherical wave* emitted from r' and the function $G_-(\vec{r} - \vec{r}\,')$ corresponds to an *incoming wave* that converges onto r .

Since the scattered waves are outgoing waves, only $G_+(\vec{r} - \vec{r}\,')$ is of interest to us.

Scattering amplitude

We are going to show here that we can obtain the differential cross section in the CM frame from an asymptotic form of the solution of the Schrödinger equation:

$$-\frac{\hbar^2}{2\mu}\vec{\nabla}^2\psi(\vec{r}) + \hat{V}(r)\psi(\vec{r}) = E\psi(\vec{r}) \qquad \text{......10}$$

❑Let us first focus on the determination of the scattering amplitude $f(\theta, \varphi)$, it can be obtained from the solutions of (10), which in turn can be rewritten as

$$(\nabla^2 + k^2)\psi(\vec{r}) = \frac{2\mu}{\hbar^2}V(\vec{r})\psi(\vec{r}). \qquad \textit{where} \quad k^2 = \frac{2\mu E}{\hbar^2} \qquad \text{......11}$$

The general solution of the equation (11) consists of a sum of two components:

1) a general solution to the homogeneous equation:

$$(\nabla^2 + k_0^2)\psi_{homo}(\vec{r}) = 0, \quad \text{where } k_0^2 = 2\mu E/\hbar^2. \qquad \text{......12}$$

In (12) $\psi_{homo}(\vec{r}) \longrightarrow \phi_{inc}(\vec{r}) = Ae^{i\vec{k}_0\cdot\vec{r}}$, is the incident plane wave

2) and a particular solution of (11) with the interaction potential

General solution of Schrödinger eq. in terms of Green's function

The general solution of (11) can be expressed in terms of Green's function.

$$\psi(\vec{r}) = \phi_{inc}(\vec{r}) + \frac{2\mu}{\hbar^2}\int G(\vec{r}-\vec{r}\,')V(\vec{r}\,')\psi(\vec{r}\,')\,d^3r', \qquad \text{……13}$$

where $\phi_{inc}(\vec{r}) = e^{i\vec{k}_0\cdot\vec{r}}$ and $G(\vec{r}-\vec{r}\,')$ is the Green's function corresponding to the operator on the left side of eq.(12)

Inserting (8) into (13) we obtain for the total scattered wave function:

This is an *integral equation.*

All we have done is to rewrite the Schrödinger (differential) equation (10) in an integral form (13), which is more suitable for scattering theory.

Note that (13) can be solved approximately by means of a series of successive or iterative approximations, known as the *Born series.*

Born Series

- the zero-order solution is given by $\psi_0(\vec{r}) = \phi_{inc}(\vec{r})$
- the first-order solution $\psi_1(\vec{r})$ is obtained by inserting $\psi_0(\vec{r}) = \phi_{inc}(\vec{r})$ into the integral of (13):

$$\begin{aligned}\psi_1(\vec{r}) &= \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} V(\vec{r}_1)\psi_0(\vec{r}_1)\, d^3r_1 \\ &= \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} V(\vec{r}_1)\phi_{inc}(\vec{r}_1)\, d^3r_1 .\end{aligned}$$

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- the second order solution is obtained by inserting $\psi_1(\vec{r})$ into (13):

$$\psi_2(\vec{r}) = \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} V(\vec{r}_2)\psi_1(\vec{r}_2)\, d^3r_2$$

$$\begin{aligned}&= \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} V(\vec{r}_2)\phi_{inc}(\vec{r}_2)\, d^3r_2 \\ &\quad + \left(\frac{\mu}{2\pi\hbar^2}\right)^2 \int \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} V(\vec{r}_2)\, d^3r_2 \int \frac{e^{ik|\vec{r}_2-\vec{r}_1|}}{|\vec{r}_2-\vec{r}_1|} V(\vec{r}_1)\phi_{inc}(\vec{r}_1)\, d^3r_1 .\end{aligned}$$

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continuing in this way, we can obtain $\psi(\vec{r})$ to any desired order

Assignment

1. Describe the method of Green's function to solve scattering Hamiltonian.
2. Explain the Green's function for point source.
3. Explain Born series expansion.

Thank You