Plasma Stability and Use of Normal Mode to Analyze Stability

Course: MPHYEC-01 Plasma Physics
(M.Sc. Sem-IV)

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Unit-03

Plasma Stability

To understand the issue of stability of equilibrium of a plasma system, we first consider a system of single particle which is under the effect of potential V(x). Further we consider the potential can have two types of shape as shown in figure below.

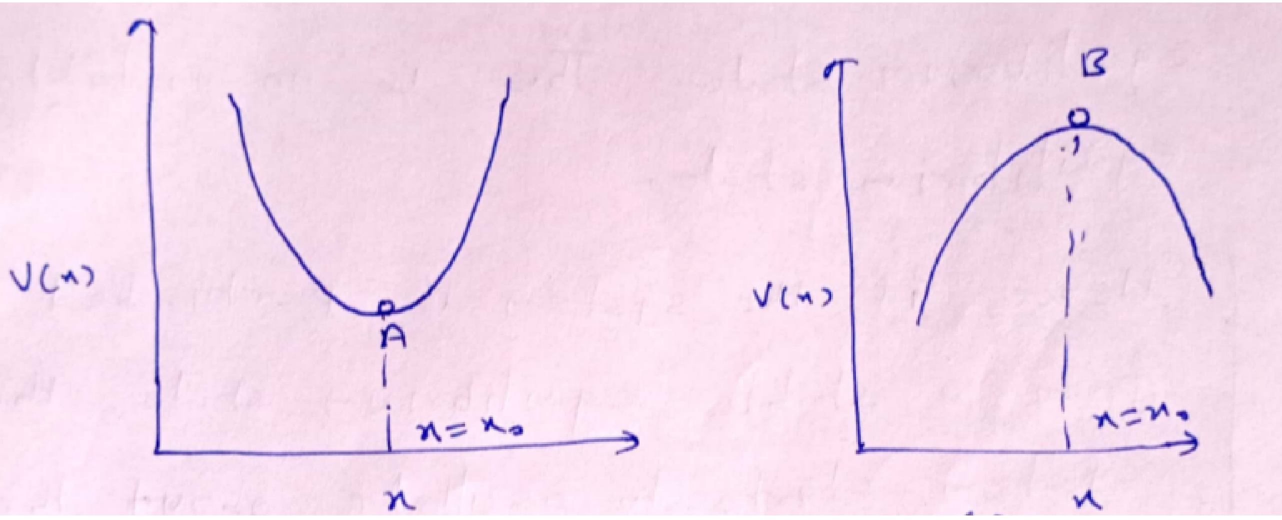


Figure 1.1 *Figure 1.2*

It is clear from the figure that particle at points A and B is in equilibrium states, as slope of V(x) at these two points is zero and hence resultant force vanishes. However both the equilibrium states are different in nature. For the equilibrium state shown in figure 1.1, if we slightly displace the particle from its equilibrium position $x=x_0$. There will be a restoring force which tries to bring the particle to the equilibrium position. Therefore, once displaced, the particle start oscillating about the equilibrium position. Such equilibrium is known as stable equilibrium. In contrast, if we consider the equilibrium shown in figure 1.2, the situation is different. If we slightly displace the particle from the equibrium position, particle will not return back to this equilibrium position. Such an equilibrium is known as unstable equilibrium.

In line of the above understanding, to examine the stability of equilibrium state of a plasma system, we introduce a slight perturbation to the equilibrium state. Now, we explore whether the perturbation oscillates with time or monotonically grows with time. If the perturbation is found to oscillate with time, we can conclude that the considered equilibrium state is stable. And if the perturbation grows with time unboundly, it means that the assumed equilibrium state is not stable. This is a general scheme to study the stability of any equilibrium state.

Normal Mode to Analyze Stability

To determine that a given equilibrium state is stable or not, one needs to carry out the normal mode analysis. For the purpose, a general strategy is following. First of all, we consider an equilibrium state. Then, we introduce a slight perturbation to disturb the equilibrium. After that, we need to linearize the equations which govern the dynamical evolution of the system. For example, if a plasma can be approximated as fluid then the governing equations for this system are MHD equations. We will only consider plasma systems which can be explained by MHD equations. After linearization of the governing MHD equations, we convert these equations in algebraic form by considering the field variables in plane wave form and then determine whether the introduced perturbation oscillates or grows with time.

Now lets consider a MHD equilibrium state characterized by:

Magnetic field $\mathbf{B} = \mathbf{B_0(r)}$
Velocity field $\mathbf{u}=\mathbf{u_0} = 0$ (static state)
Mass density $\rho = \rho_0(r)$
Pressure $p = p_0(r)$

Note that the field variables are time-independent but generally space-dependent. From the momentum transport equation, a MHD equilibrium state is explained by following equation:

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B}$$

The equation for the considered equilibrium state modifies as:

$$\nabla p_0 = \frac{(\nabla \times \boldsymbol{B_0})}{\mu_0} \times \boldsymbol{B_0} \qquad \text{---------} \ (1)$$

Further lets consider that we introduce a small perturbation and, as a result, the equilibrium state is perturbed as:

$\mathbf{B} = \mathbf{B_0(r)} + \mathbf{B_1(r}, t\mathbf{)}$
$\mathbf{u} = 0 + \mathbf{u_1(r}, t\mathbf{)}$
$\rho = \rho_0(r) + \rho_1(r, t)$
$p = p_0(r) + p_1(r, t)$

where the change in field variables due to the perturbation is denoted by 1 subscripts. As the perturabtion is small, the magnitude of the change is also small. Next, we would like to linearize the governing MHD equations.

First, we linearize mass continuity equation by neglecting multiplication of two perturbing quantities:

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho \boldsymbol{u})=0$$

$$\frac{\partial(\rho_0+\rho_1)}{\partial t}+\nabla\cdot((\rho_0+\rho_1)\boldsymbol{u_1})=0$$

$$\frac{\partial \rho_1}{\partial t}+\nabla\cdot(\rho_0 \boldsymbol{u_1})=0$$

$$\frac{\partial \rho_1}{\partial t}+\nabla \rho_0\cdot\boldsymbol{u_1}+\rho_0\nabla\cdot\boldsymbol{u_1}=0 \qquad \text{---------- (2)}$$

This is the linearized mass continuity equation.

Next, we linearize momentum transport equation:

$$\rho\frac{d\boldsymbol{u}}{dt}=-\nabla p+\boldsymbol{J}\times\boldsymbol{B}$$

$$(\rho_0+\rho_1)(\frac{\partial \boldsymbol{u_1}}{\partial t}+\boldsymbol{u_1}\cdot\nabla\boldsymbol{u_1})=-\nabla(p_0+p_1)+\frac{\nabla\times(\boldsymbol{B_0}+\boldsymbol{B_1})}{\mu_0}\times(\boldsymbol{B_0}+\boldsymbol{B_1})$$

Neglecting the terms which contain two perturbing quantities and using equation (1), we can obtain:

$$\rho_0\frac{\partial \boldsymbol{u_1}}{\partial t}=-\nabla p_1+\frac{\nabla\times\boldsymbol{B_0}\times\boldsymbol{B_1}}{\mu_0}+\frac{\nabla\times\boldsymbol{B_1}\times\boldsymbol{B_0}}{\mu_0} \qquad \text{---------- (3)}$$

This is the linearized momentum transport equation.

Further, we need to linearize the induction equation which is given as

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})$$

Note that we have assumed the plasma to be perfectly electrical conducting and, therefore, neglected the diffusion term in the induction equation.

$$\frac{\partial(\boldsymbol{B_0}+\boldsymbol{B_1})}{\partial t}=\nabla\times(\boldsymbol{u_1}\times(\boldsymbol{B_0}+\boldsymbol{B_1}))$$

$$\frac{\partial \boldsymbol{B_1}}{\partial t} = \nabla \times (\boldsymbol{u_1} \times \boldsymbol{B_0}) \qquad \text{---------} \ (4)$$

This is the linearized induction equation.

Finally, we are left with pressure which is specified by the equation of state,

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right) + \boldsymbol{u} \cdot \nabla \left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Further manipulations and use of the mass continuity equation modifies the above equation as:

$$\frac{\partial p}{\partial t} + \gamma p \nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla p = 0$$

Now,

$$\frac{\partial (p_0 + p_1)}{\partial t} + \gamma (p_0 + p_1) \nabla \cdot \boldsymbol{u_1} + \boldsymbol{u_1} \cdot \nabla (p_0 + p_1) = 0$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0 \nabla \cdot \boldsymbol{u_1} + \boldsymbol{u_1} \cdot \nabla p_0 = 0 \qquad \text{----------} \ (5)$$

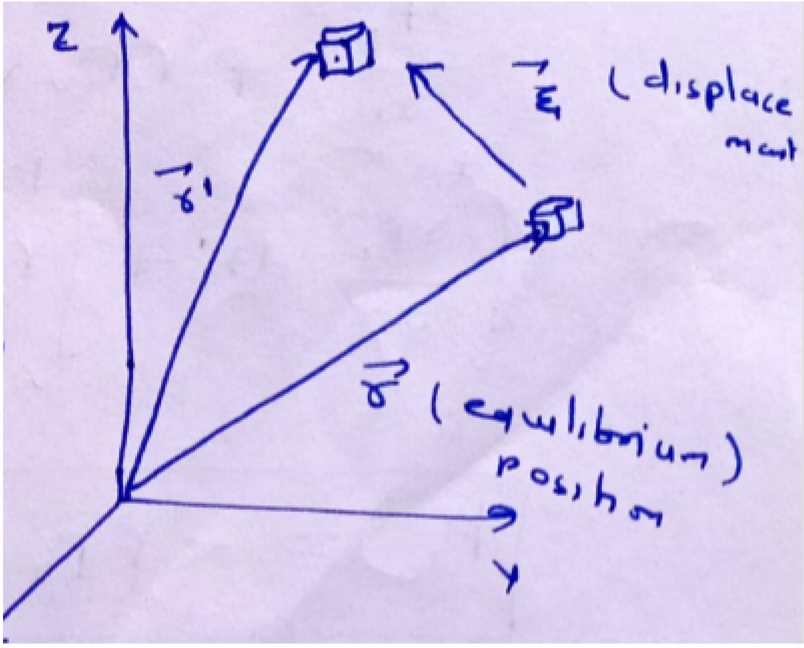
This is the linearized equation of state.

Note that the equations (2)-(4) determine the response of the system to the introduced perturbation. Now, for the normal mode analysis, lets assume that a fluid parcel is situated at position **r** when plasma is in equilibrium state, as shown in figure below. After the perturbation, the parcel is displaced at position $\mathbf{r}'$. Then we can define a displacement vector $\boldsymbol{\xi}$ which determines how a fluid parcel gets displaced from its equilibrium position after the introduction of perturbation.

Then the velocity vector field $\mathbf{u_1}$ can be defined as:

$$\boldsymbol{u_1} = \frac{\partial \boldsymbol{\xi}(\boldsymbol{r}, t)}{\partial t} \qquad \text{--------} \ (6)$$

$$\boldsymbol{\xi}(\boldsymbol{r}, t) = \int_0^t \boldsymbol{u_1}(\boldsymbol{r}, t) \, dt'$$



Using equation (6) into equation (2), we get

$$\frac{\partial \rho_1}{\partial t}+\frac{\partial \boldsymbol{\xi}}{\partial t}\cdot\nabla\rho_0+\rho_0\nabla\cdot\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right)=0$$

Integrating the equation with time, we get

$$\rho_1+\boldsymbol{\xi}\cdot\nabla\rho_0+\rho_0\nabla\cdot\boldsymbol{\xi}=0$$

$$\rho_1=-\boldsymbol{\xi}\cdot\nabla\rho_0-\rho_0\nabla\cdot\boldsymbol{\xi} \qquad \text{---------} \ (7)$$

Using equation (6) into equation (4), we obtain

$$\frac{\partial \boldsymbol{B_1}}{\partial t}=\nabla\times\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\times\boldsymbol{B_0}\right)$$

Integrate the above equation with time,

$$\boldsymbol{B_1}=\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B_0}) \qquad \text{---------} \ (8)$$

Now from equation (5), we have

$$\frac{\partial p_1}{\partial t}+\gamma p_0\nabla\cdot\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right)+\frac{\partial \boldsymbol{\xi}}{\partial t}\cdot\nabla p_0=0$$

and integration with time leads to

$$p_1 + \gamma\, p_0 \nabla \cdot \xi + \xi \cdot \nabla\, p_0 = 0$$

$$p_1 = -\gamma\, p_0 \nabla \cdot \xi - \xi \cdot \nabla\, p_0 \qquad \text{-----------} \; \mathbf{(9)}$$

Put ρ_1, $\mathbf{B_1}$, p_1 from equations (7), (8) and (9) into equation (3), we can obtain the following governing equation for **ξ**,

$$\rho_0 \frac{\partial^2 \xi(\boldsymbol{r}, t)}{\partial t^2} = \boldsymbol{F}(\xi) \qquad \text{----------- (10)}$$

where

$$\boldsymbol{F}(\xi) = \nabla(\xi \cdot \nabla\, p_0 + \gamma\, p_0 \nabla \cdot \xi) + \frac{1}{4\pi}(\nabla \times \boldsymbol{B_0}) \times [\nabla \times (\xi \times \boldsymbol{B_0})] + \frac{1}{4\pi}[\nabla \times \nabla \times (\xi \times \boldsymbol{B_0}) \times \boldsymbol{B_0}]$$

is known as MHD force operator.

In principle, from equation (10), we can solve for **ξ** and determine whether **ξ . F > 0** or **ξ . F <0.**

The case of **ξ . F <0** implies that the displacement and force are in opposite direction. Hence, due to this restoring nature of the force, the fluid parcel oscillates around the equilibrium position. In other words, the system oscillates around the equilibrium state and this equilibrium state is stable. In contrast, if **ξ . F > 0**, the displacement and force is in same direction. As a result, the fluid parcel monotonically displaces from the equilibrium position. In other words, the system doesn't return back to the equilibrium state.

Furthermore, to determine the normal mode, we utilize the following expression of **ξ,**

$$\xi(\boldsymbol{r}, t) = \xi_{\mathbf{0}}(\boldsymbol{r})\, e^{-i\omega t}$$

and if we insert this expression in equation (10), we can obtain ω. Now, if ω is a real number then $\xi(\boldsymbol{r}, t) = \xi_{\mathbf{0}}(\boldsymbol{r})\, e^{-i\omega t}$ will be an oscillatory function and the corresponding equilibrium will be stable. If ω is a positive imaginary number then $\xi(\boldsymbol{r}, t) = \xi_{\mathbf{0}}(\boldsymbol{r})\, e^{-i\omega t}$ will grow with time and the corresponding equilibrium will be unstable and give rises instability.

Thanks for the attention!