Relativistic Electrodynamics
M.Sc. 2nd Semester
MPHYCC-6: Electrodynamics and Plasma Physics
Unit IV
Topic: Maxwell's equations in Four-Tensor notation

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Maxwell's equations in Four-tensor notation

In four dimensional a two-index antisymmetric tensor has (4 × 3)/2 = 6 independent components. Since this is equal to 3 + 3, it suggests that perhaps we should be grouping the electric and magnetic fields together into a single 2-index antisymmetric tensor. Thus we introduce a tensor $F_{\mu\nu}$, satisfying

$$F\mu\nu = -F\nu\mu \tag{1}$$

It turns out that we should define its components in terms of $\vec{E}$ and $\vec{B}$ as follows:

$$F_{0i} = -E_i\,, \qquad F_{i0} = E_i\,, \qquad F_{ij} = \epsilon_{ijk}\,B_k \tag{2}$$

Here ε_{ijk} is the usual totally-antisymmetric tensor of 3-dimensional vector calculus. It is equal to +1 if (ijk) is an even permutation of (123), to = −1 if it is an odd permutation, and to zero if it is no permutation (i.e. if two or more of the indices (ijk) are equal). In other words, we have

$$\begin{aligned} F_{23} &= B_1\,, \qquad F_{31} = B_2\,, \qquad F_{12} = B_3\,, \\ F_{32} &= -B_1\,, \qquad F_{13} = -B_2\,, \qquad F_{21} = -B_3 \end{aligned} \tag{3}$$

Viewing $F_{\mu\nu}$ as a matrix with rows labelled by μ and columns labelled by ν, we shall have

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{4}$$

We also need to combine the charge density ρ and the 3-vector current density $\vec{J}$ into a four-dimensional quantity. We define a four-vector J_μ, whose spatial components J_i are just the usual 3-vector current components, and whose time component J_0 is equal to the charge density ρ:

$$J^0 = \rho\,, \qquad J^i = J^i\,. \tag{5}$$

Maxwell equations expressed in terms of $F_{\mu\nu}$ and J_μ

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -4\pi J^\nu\,, \\ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} &= 0\,. \end{aligned} \tag{6}$$

The equations are manifestly Lorentz covariant, i.e. they transform like tensor under Lorentz transformations. This means that they keep the same form in all Lorentz frames. This equation is vector-valued, since it has the free index ν. Therefore, to reduce it down to three-dimensional equations, we have two cases to consider, namely ν = 0 or ν = j. For ν = 0 we have

$$\partial_i F^{i0} = -4\pi J^0 \tag{7}$$

which therefore corresponds to

$$-\partial_i E_i = -4\pi\rho\,, \qquad \text{i.e.} \qquad \vec{\nabla}\cdot\vec{E} = 4\pi\rho\,. \tag{8}$$

For ν = j, we shall have

$$\partial_0 F^{0j} + \partial_i F^{ij} = -4\pi J^j \tag{9}$$

which gives

$$\partial_0 E_j + \epsilon_{ijk}\partial_i B_k = -4\pi J^j \tag{10}$$

This is just

$$-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = 4\pi \vec{J}\,. \tag{11}$$

Turning now to Eqn. (6), it follows from the antisymmetric Eqn. (1) of $F_{\mu\nu}$ that the left-hand side is totally antisymmetric in $(\mu\nu\rho)$ (i.e. it changes sign under any exchange of a pair of indices). Therefore, there are two distinct assignments of indices, after we make the 1 + 3 decomposition $\mu = (0, i)$ etc. Either one of the indices is a 0 with the other two Latin, or else all three are Latin. Consider first $(\mu, \nu, \rho) = (0, i, j)$:

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0 \tag{12}$$

which, from Eqn. (2), means

$$\epsilon_{ijk}\frac{\partial B_k}{\partial t} + \partial_i E_j - \partial_j E_i = 0 \tag{13}$$

Since this is antisymmetric in ij there is no loss of generality involved in contracting with $\varepsilon_{ij\ell}$, which gives

$$2\frac{\partial B_\ell}{\partial t} + 2\epsilon_{ij\ell}\,\partial_i E_j = 0 \tag{14}$$

This is just the statement that

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{15}$$

Which is one of the Maxwell's equation. The other distinct possibility for assigning decomposed indices in Eqn. (6) is to take $(\mu, \nu, \rho) = (i, j, k)$, giving

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0 \tag{16}$$

Since this is totally antisymmetric in (i, j, k), no generality is lost by contracting it with ε_{ijk}, giving

$$3\epsilon_{ijk}\,\partial_i F_{jk} = 0 \tag{17}$$

Which implies

$$3\epsilon_{ijk}\epsilon_{jk\ell}\partial_i B_\ell = 0\,, \qquad \text{and hence} \qquad 6\partial_i B_i = 0 \tag{18}$$

This has just reproduced the Maxwell equation in $\vec{\nabla} \cdot \vec{B} = 0$.

We may begin by considering the quantities $J^\mu = (\rho, J^i)$. Note first that by applying ∂_ν to the Maxwell field equation (3), we get identically zero on the left-hand side, since partial derivatives commute and $F^{\mu\nu}$ is antisymmetric. Thus, from the left-hand side we get

$$\partial_\mu J^\mu = 0 \tag{19}$$

This is the equation of charge conservation. Decomposed into the 3 + 1 language, it takes the familiar form

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{20}$$

By integrating over a closed 3-volume V and using the divergence theorem on the second term, we learn that the rate of change of charge inside V is balanced by the flow of charge through its boundary S:

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_S \vec{J}\cdot d\vec{S} \tag{21}$$

Now we are in a position to show that $J^\mu = (\rho, \vec{J})$ is indeed a four vector. Considering $J^0 = \rho$ first, we may note that

$$dQ \equiv \rho dx dy dz \tag{22}$$

is clearly Lorentz invariant, since it is an electric charge. Clearly, for example, all Lorentz observers will agree on the number of electrons in a given closed spatial region, and so they will agree on the amount of charge. Another quantity that is Lorentz invariant is dv = dtdxdydz, the volume of an infinitesimal region in spacetime. This can be seen from the fact that the Jacobian J of the transformation from dv to dv′ = dt′dx′dy′dz′ is given by

$$\mathcal{J} = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \det(\Lambda^\mu{}_\nu) \tag{23}$$

the Lorentz transformation can be written in a matrix notation as $\Lambda^T\eta\,\Lambda = \eta$ and hence taking the determinant, we get $(\det \Lambda)^2 = 1$ and hence $\det \Lambda = \pm 1$. If we restrict attention to Lorentz transformations without reflections, then they will be connected to the identity, and so $\det \Lambda = 1$. Thus, it follows from Eqn. (23) that for Lorentz transformations without reflections, the four-volume element dtdxdydz is Lorentz invariant. Comparing dQ = ρdxdydz and dv = dtdxdydz, both of which we have argued are Lorentz invariant, we can conclude that ρ must transform in the same way as dt under Lorentz transformations. In other words, ρ must transform like the 0 component of a four-vector. Thus writing, as we did, that $J^0 = \rho$, is justified. In the same way, we may consider the spatial components J^i of the putative four-vector J^μ. Considering J^1, for example, we know that $J^1 dydz$ is the current flowing through the area element dydz. Therefore, in time dt, there will have been a flow of charge $J^1 dtdydz$. Being a charge, this must be Lorentz invariant, and so it follows from the known Lorentz invariance of dv = dtdxdydz that J^1 must transform the same way as dx under Lorentz transformations. Thus, J^1 does indeed transform like the 1 component of a four-vector. Similar arguments apply to J^2 and J^3. We have now established that $J^\mu = (\rho, J^i)$ is indeed a Lorentz four-vector, where ρ is the charge density and J^i the 3-vector current density.

Using Lorenz gauge

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{\partial t} = 0 \tag{24}$$

We can write Maxwell's field equation as:

$$\Box A^\mu = -4\pi\, J^\mu\,,$$

where

$$A^\mu = (\phi, \vec{A})\,, \tag{25}$$

We saw that it is manifestly a Lorentz scalar operator, since it is built from the contraction of indices on the two Lorentz-vector gradient operators. Since we have already established that J^μ is a four-vector, it therefore follows that A^μ is a four-vector. The Lorenz gauge condition translates, in the four-dimensional, into

$$\partial_\mu A^\mu = 0 \tag{26}$$

If we write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{27}$$

The $\eta_{00} = -1$ when lowering the 0 index, shall give

$$A_\mu = (-\phi, \vec{A}) \tag{28}$$

Therefore, we find

$$\begin{aligned} F_{0i} &= \partial_0 A_i - \partial_i A_0 = \frac{\partial A_i}{\partial t} + \partial_i \phi = -E_i\,, \\ F_{ij} &= \partial_i A_j - \partial_j A_i = \epsilon_{ijk} (\vec{\nabla} \times \vec{A})_k = \epsilon_{ijk}\, B_k \end{aligned} \tag{29}$$

we have shown that J^μ is a four-vector, and hence, A^μ is a 4-vector. Then, it is that $F_{\mu\nu}$ is a four-tensor. Hence, we have established that the Maxwell equations, are indeed expressed in terms of four-tensors and four-vectors, and so the manifest Lorentz covariance of the Maxwell equations is established.

Reference:

1. http://people.physics.tamu.edu/pope/EM611/em611-2010.pdf