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Solution to the Dirac equation

(Pauli-Dirac representation)

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Advanced Quantum Mechanics (EC 01)

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Dirac equation and its solution
(Pauli − Dirac representation)

Dirac equation is given by

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0. \qquad (1)$$

As explained in the classroom we follow the convention (Pauli − Dirac representation for Clifford algebra) which is based on the Pauli spin matrices and write the gamma matrices as:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \qquad (2)$$

These matrices satisfy the Clifford algebra $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$. We rewrite the Dirac equation in the form similar to that of Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi = c\left(\vec{\alpha}.\vec{P} + mc\beta\right)\psi, \qquad (3)$$

where $\vec{P} = -i\hbar\vec{\nabla}$ (to be distinguished with c-number $\vec{P}$).We take $\hbar = c = 1$ as usual. The matrices $\vec{\alpha}$ *and* β are defined by

$$\beta = \gamma^0, \qquad \vec{\alpha} = \gamma^0\vec{\gamma}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \qquad (4)$$

In general , for momentum

$$\vec{P} = p(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \textit{(spherical coordinates)}$$

we define two-component eigen-states of the matrix $\vec{\sigma}.\vec{P}$ as :

$$\chi + (\vec{p}) = \begin{pmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \qquad (5)$$

$$\chi - (\vec{p}) = \begin{pmatrix} -sin\frac{\theta}{2}e^{-i\phi} \\ cos\frac{\theta}{2} \end{pmatrix}, \qquad (6)$$

Which satisfy

$$(\vec{\alpha}.\vec{p})\chi \pm (\vec{p}) = \pm p\chi \pm (\vec{p}). \qquad (7)$$

Using $\chi\pm$, we can write down solutions to the Dirac equation.

The Positive energy solutions with momentum $\vec{p}$ have space and time dependence $\psi_\pm(x,t) = u_\pm(p)e^{-iEt+i\vec{p}\cdot\vec{x}}$.

The subscript ± refers to the helicities ±1⁄2. The Dirac equation then reduces to an equation with no derivatives:

$$E\psi = (\alpha\cdot\vec{p} + m\beta)\psi \tag{8}$$

where $\vec{p}$ is the momentum vector (not an operator). The explicit solutions can be obtained easily as

$$u_+(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix}(E+m)\chi_+(\vec{p})\\ p\chi_+(\vec{p})\end{pmatrix}, \tag{9}$$

$$u_-(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix}(E+m)\chi_-(\vec{p})\\ -p\chi_-(\vec{p})\end{pmatrix} \tag{10}$$

We adoptthe normalization $u_\pm^\dagger(p)u_\pm(p) = 2E \text{ and } E = \sqrt{\vec{p}^2+m^2}$.

Negative energy solutions must be filled in the vacuum and their "holes" are regarded as anti-particle states. Therefore, it is convenient to assign momentum $-\vec{p}$ and energy $-E = -\sqrt{\vec{p}^2+m^2}$.

The solutions have space and time dependence $\psi_\pm(x,t) = v_\pm(p)e^{+iEt-i\vec{p}.\vec{x}}$.The Dirac equation again reduces to an equation with no derivatives:

$$-E\psi = (-\alpha\cdot\vec{p} + m\beta)\psi. \tag{11}$$

Explicit solutions are given by

$$v_+(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix}-p\chi_-(\vec{p})\\ (E+m)\chi_-(\vec{p})\end{pmatrix}, \tag{12}$$

$$v_-(p) = \frac{1}{\sqrt{E+m}}\begin{pmatrix}p\chi_+(\vec{p})\\ (E+m)\chi_+(\vec{p})\end{pmatrix}, \tag{13}$$

It is convenient to define "barred" spinors $\bar{u} = u^\dagger\gamma^0 = u^\dagger$ and $\bar{v} = v^\dagger\gamma^0$. The combination $\bar{u}u$ is a Lorentz-invariant, $\bar{u}u = 2m$, and similarly, $\bar{v}v = -2m$. The combination $\bar{u}\gamma^\mu u$ transforms as a Lorentz vector:

$$\bar{u}_\kappa(p)\gamma^\mu u_\lambda(p) = 2p^\mu\delta_{\kappa,\lambda}, \tag{14}$$

where $\kappa, \lambda = \pm$, and similarly,

$$\bar{v}_\kappa(p)\gamma^\mu v_\lambda(p) = 2p^\mu\delta_{\kappa,\lambda}. \tag{15}$$

They can be interpreted as the "four-current density" which generates electromagnetic field: $\bar{u}\gamma^0 u = u^\dagger u$ is the "charge density," and $\bar{u}\gamma^i u = u^\dagger \alpha^i u$ is the "current

Density".

Further the matrix

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

commutes with the Hamiltonian in the massless limit $m \to 0$. In fact, at high energies $E\ m$, the solutions are almost eigenstates of γ_5, with eigenvalues +1 for u_+ and v_-, and 1 for u_- and v_+. The eigenvalue of γ_5 is called "chirality." Therefore chirality is a good quantum number in the high energy limit. Neutrinos have chirality minus so they do not have states with positive chirality.
The diagrams for positive and negative energy states is given in reference number (2).
Doubts can be cleared in our Whatsapp group.

REFERENCES:

- 1.Quantum Mechanics by G. Aruldhas
- 2.Relativistic Quantum Mechanics by James D. Bjorken/Sidney D. Drell
- 3.Modern particle physics by Mark Thomson
- 4.Quantum Mechanics by V.K. Thankappan
- 5.Advanced quantum mechanics by J.J.Sakurai.