Matrix Inversion: Determinant Method

MPHYCC-05 unit IV (Sem.-II)

Why inversion of a matrix?

We know that the solving the systems of linear equations is one of the solid workhorses of numeric computing. It used everywhere from *geometry* e.g. graphics, games, navigation, to *modeling physical systems* e.g. weather simulation, fluid dynamics, chemical reactions, to *statistical analysis*, and beyond.

And for solving the set of linear equations in easy way, we write them in in a matrix form. However, we cannot do the division with matrix therefore we use inverse of matrix for the division. For example: *If you have to divide 20 rupees in 4 people then you can simply do 20/4=5 or you can find the inverse of 4 that is 1/4 and multiply it with 20 which gives (20*1/4 =5).* Similarly if we have to calculate value of a matrix **X** which follow the equation **AX=B.** The value of **X=B\A** but we cannot do the matrix division; however we can multiply **B** with the inverse of A. So this can be written as $\mathbf{X = (A^{-1})*B}$. Following is the example of a set of linear equations;

$$a_{11}\, x_1 + a_{12}\, x_2 + ... + a_{1n}\, x_n = b_1$$
$$a_{21}\, x_1 + a_{22}\, x_2 + ... + a_{1n}\, x_n = b_2$$
....
....
....
$$a_{n1}\, x_1 + a_{n2}\, x_2 + ... + a_{nn}\, x_n = b_n$$

The above set of equation can be written in the matrix form as follows.

$$AX=B$$

Where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

In-order to have the solution that is X = (A^{-1})*B which required to calculate the inverse of the matrix A. Here, we discuss about the determinant method to compute the inverse of a matrix.

Determinant method

In number arithmetic every number b (≠ 0) has an inverse say c and written as b^{-1}. It can also be written as bc = cb = 1. However, in case of matrix not all the square matrices are having the inverses only some of them. If a square matrix with detA≠0 has an inverse, A^{-1}, then it must follow the relation $AA^{-1} = A^{-1}A = I$. It should be noted that the non-square matrices do not have inverses. Moreover, the matrices having inverses are known as non-singular (no-degenerate) otherwise singular (degenerate).

To obtain the A^{-1} using determinant methods the following steps has to be taken. For a given square matrix A with detA≠0:

- Find |A|. If |A| = 0 then A^{-1} does not exist. If |A| ≠ 0 then it will be possible to calculate the inverse of matrix, as follows.
- Replace each element of A by its cofactor.
- Transpose the result to form the adjoint matrix, denoted by adj(A)
- Then calculate $A^{-1} = \text{adj}(A)/|A|$.

First we learn how to find out the cofactor of a matrix. The cofactor is defined as the signed *minor*. An cofactor corresponding to the (i,j) positioned matrix element computed by multiplying minor of the (i,j) element by $(-1)^{i+j}$ and is denoted by C_{ij}. And the formula to find cofactor, $C_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} denotes the minor of an element of a matrix having position (i^{th} *row* and j^{th} column). And the minor is defined as a value obtained from the determinant of a square matrix by deleting out a row and a column corresponding to the element of a matrix. Following steps are required to be applying in-order to compute the minor of a matrix.

- Hide the i^{th} row and j^{th} column one by one from given matrix, where **i** refer to **m** and **j** refers to **n** that is the total number of rows and columns in matrices, respectively.
- Evaluate the value of the determinant of the matrix made after hiding a row and a column from Step 1.

Minors can be calculated using the above steps, then after applying the required signed to minors; one can form the cofactor matrix. And transposing the cofactor matrix will result into an adjoint matrix. After calculating the determinant of the matrix and with the help of the respective adjoint matrix, we can easily compute the inverse of a matrix as described above.

Followings are some examples demonstrating the determinant method to compute the inverse of a 2x2 and 3x3 square matrices.

Example 1: Find the inverse of the following matrix.

$$A = \begin{bmatrix} 5 & -7 \\ -10 & 14 \end{bmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 5 & -7 \\ -10 & 14 \end{vmatrix} = 14*5-(-7)*(-10) = 0$$

Therefore, A^{-1} does not exist.

Example 2: Find the inverse of the matrix $A = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}$

Solution:

$$|A| = \begin{vmatrix} 2 & -2 \\ 1 & -2 \end{vmatrix} = (2)*(-2) - (1)*(-2) = -2$$

Therefore, A^{-1} does exist.

The minor (M_{ij}) and cofactor (C_{ij}) of the element are:

$M_{11} = -2$ $\quad C_{11} = -2$

$M_{12} = 1$ $\quad C_{12} = -1$

$M_{21} = -2$ $\quad C_{21} = 2$

$M_{22} = 2$ $\quad C_{22} = 2$

$$\text{Cofactor matrix} = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$$

$$\text{and adj } A = \begin{bmatrix} -2 & 2 \\ -1 & 2 \end{bmatrix}$$

A^{-1} = adj A/|A| , therefore

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 1/2 & -1 \end{bmatrix}$$

Example 3: Find the inverse of the following matrix.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$$

Solution:

Determinant of A is $|A| = 0*(3+2) - 1*(2-1) + 1*(4+3) = 6$

$|A| \neq 0$, thus A^{-1} exist

Matrix made up of the cofactors:

$$\begin{bmatrix} +\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 7 \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{bmatrix}$$

$$\text{Thus } \operatorname{Adj}(A) = \begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$$

$$\text{Thus } A^{-1} = \operatorname{Adj}(A)/|A| = \frac{1}{6}\begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$$

Working with determinant method is easy for the matrix with lower dimension but difficult for higher dimensional matrix. In principle, the inverse of a matrix with higher dimension could be found with same method; however, the process is more tedious and takes longer.

Python codes are given below to find the inverse of a 2 by 2 matrix using determinant method.

```
# Python code to get inverse of a 2 by 2 matrix m
# define the matrix transpose (MT)
def MT(m):
    return map(list,zip(*m))

# define and calculate the minor of the matrix (MM)
def MM(m,i,j):
```

```
    return [row[:j] + row[j+1:] for row in (m[:i]+m[i+1:])]

# define and calculate the determinant of the matrix
def MD(m):
    return m[0][0]*m[1][1]-m[0][1]*m[1][0]

# define and calculate the inverse of the matrix (MI)
def MI(m):
    determinant = MD(m)
    return [[m[1][1]/determinant, -1*m[0][1]/determinant],
            [-1*m[1][0]/determinant, m[0][0]/determinant]]

#find matrix of cofactors
    cofactors = []
    for r in range(len(m)):
        cofactorRow = []
        for c in range(len(m)):
            minor = MM(m,r,c)
            cofactorRow.append(((-1)**(r+c)) * MD(minor))
        cofactors.append(cofactorRow)
    cofactors = MT(cofactors)
    for r in range(len(cofactors)):
        for c in range(len(cofactors)):
            cofactors[r][c] = cofactors[r][c]/determinant
    return cofactors
```

After compiling the above python script and for the different matrices (m) we have the following out puts:

```
>>> import numpy as np
>>> m=np.array([[2,-2],[1,-2]])
>>> MI(m)
[[1.0, -1.0], [0.5, -1.0]]

>>> m=np.array([[-2,-2],[1,-2]])
>>> MI(m)
[[-0.3333333333333333, 0.3333333333333333],
 [-0.16666666666666666, -0.3333333333333333]]
```

```
>>> m=np.array([[-2,2],[-1,2]])
>>> MI(m)
[[-1.0, 1.0], [-0.5, 1.0]]

>>> m=np.array([[2,3],[5,6]])
>>> MI(m)
[[-2.0, 1.0], [1.6666666666666667, -0.6666666666666666]]
```

However, the inverse of a matrix can be easily found with the help of ***numpy*** that is a kind of python libraries. The required Python script using numpy is given as follows.

```
import numpy as np
row = int(input("Enter the number of rows:"))
column = int(input("Enter the number of columns:"))
print("Enter the matrix element separated by space: ")
if(row == column):

# User input of entries in a single line separated by space
    entries = list(map(int, input().split()))

# Forming the required matrix from the inputs
    matrixm = np.array(entries).reshape(row, column)

# Inverse of matrix using the attribute inv included
# in linear algebra (linalg) package of numpy
    matrixinverse = np.linalg.inv(matrixm)
    print('Inverse of the matrix:\n', matrixm, '\n is \n',matrixinverse)
else:
   print("For inverse calculation matrix should be square\n thus enter the same number of rows and columns")
```

After compiling the above python script and for the different matrices we have the following out puts:

…………………………………………………………..
Enter the number of rows:2
Enter the number of columns:2
Enter the matrix element separated by space:
2 3 4 5
Inverse of the matrix:
 [[2 3]
 [4 5]]
 is
 [[-2.5 1.5]
 [2. -1.]]

……………………………………………………………..
Enter the number of rows:3
Enter the number of columns:3
Enter the matrix element separated by space:
4 2 -2 1 -3 -1 3 -1 4
inverse of the matrix:
 [[4 2 -2]
 [1 -3 -1]
 [3 -1 4]]
 is
 [[0.15853659 0.07317073 0.09756098]
 [0.08536585 -0.26829268 -0.02439024]
 [-0.09756098 -0.12195122 0.17073171]]

……………………………………………………………….
Enter the number of rows:2
Enter the number of columns:3
Enter the matrix element separated by space:
For inverse calculation matrix should be square
 thus enter the same number of row and column

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