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# Klein Gordon equation and the concept of positive and negative probability density values

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**Klein Gordon equation and the concept of positive and negative probability density values.**

In non-relativistic case the position probability density $\rho(r,t)$ is defined as $|\psi(r,t)|^2$and the probability current density j (r, t).

$\rho(r,t) and$ j (r, t) satisfy the equation of continuity, which is invariant under Lorentz transformation.

The relativistic Schrodinger equation is

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$$\frac{-\hbar^2 \partial^2 \psi(r,t)}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi(\vec{r},t) + m_0^2 c^4 \, \psi(\vec{r},t) \qquad \ldots\ldots (1)$$

The non- relativistic continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla . \vec{j} = 0$$

$\rho(\vec{r}) = |\psi|^2$ probability density

$$\vec{j} = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*)$$

The probability current density is

$$\vec{j} = \frac{i\hbar}{2m} ( \psi \nabla \psi^* - \psi^* \nabla \psi)$$

The complex conjugate of relativistic Schrodinger equation is

$$\frac{-\hbar^2 \partial^2 \psi^*}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi^* + m_0^2 c^4 \, \psi^* \qquad \ldots\ldots. (2)$$

Now, multiply equation (3) by $\psi^*$ and (2) by $\psi$ and then subtracting one from the other

$$-\hbar^2 \left( \psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} \right) = -\hbar^2 c^2 [\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*]$$

$$\frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) = c^2 \nabla[\psi^* \nabla \psi - \psi \nabla \psi^*]$$

$$\frac{i\hbar}{2mc^2}\frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) = \frac{i\hbar}{2m} \nabla[\psi^* \nabla \psi - \psi \nabla \psi^*]$$

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2mc^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right] + \nabla\left[\frac{i\hbar}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi)\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla . \vec{j} = 0$$

$$\therefore \qquad \vec{j} = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*)$$

$$\vec{j} = \frac{i\hbar}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi)$$

$$\therefore \quad \rho = \frac{i\hbar}{2mc^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)$$

$$= \frac{1}{2mc^2}\left[\psi^* i\hbar \frac{\partial \psi}{\partial t} + \psi(-i\hbar) \frac{\partial \psi^*}{\partial t}\right]$$

$$= \frac{1}{2mc^2}[\psi^* E \psi + \psi E \psi^*]$$

$$\rho = \frac{2E|\psi|^2}{2mc^2}$$

$$\boldsymbol{\rho = \frac{E}{mc^2}|\psi|^2 \quad \ldots\ldots. (3)}$$

From equation (3), it follows that $\rho(r,t)$ is positive when E is positive and negative when E is negative. In other word the probability density takes both positive and negative values.

Pauli and Weisskopf interpreted $q\rho$ as the electrical charge density and $\mathbf{q}\mathbf{j}$ as the corresponding electric current density. This is reasonable as charges can take positive or negative values. If the system has a single particle of given charge, $\boldsymbol{\rho}$ cannot have different signs at different point. This means that theory is useful only to a system of particles having both signs of theory is useful only to a system of particles having both signs of charges.

For $\rho > 0 \Rightarrow E > 0$ positive charge particles

$\rho < 0 \Rightarrow E < 0$ Antiparticles (negative charge)

This definition of $\rho(r,t)$ leads to both positive and negative value for it.KG equation is used for a system of particles having both positive and negative charges.

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