M PHY CC-06

Propagation of plane E.M.W. in Free Space

[Unit I]

For free space
$\rho = 0,\ \sigma = 0,\ \mu = \mu_0$ and $\epsilon = \epsilon_0$

So, Maxwell's eqn is

$$\text{div } \vec{D} = \vec{\nabla}\cdot\vec{D} = 0 \quad \text{— (i)}$$

$$\text{div } \vec{B} = \vec{\nabla}\cdot\vec{B} = 0 \quad \text{— (ii)}$$

$$\text{curl } \vec{E} = \vec{\nabla}\times\vec{E} = (-)\frac{\partial B}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad \text{— (iii)}$$

and,
$$\text{curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (\text{Since } \sigma = 0) \quad \text{— (iv)}$$

With $\vec{J} = \sigma\vec{E}$, $\vec{B} = \mu\vec{H}$, $\vec{D} = \epsilon\vec{E}$

Taking curl of Eqn (iii)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

$$\Rightarrow \nabla(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu_0\epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial \vec{E}}{\partial t}\right) \quad \text{from (iv)}$$

from (i) $\vec{\nabla}\cdot\vec{E} = 0$, $c^2 = \frac{1}{\mu_0\epsilon_0}$

$$\therefore \boxed{\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0} \quad \text{— (A)}$$

Similarly Taking Curl of Eqn (iv)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \epsilon_0\left(\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow \nabla(\vec{\nabla}\cdot\vec{H}) - \nabla^2 H = \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = -\epsilon_0\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$-\nabla^2\vec{H} = -\frac{1}{c^2}\frac{\partial \vec{H}}{\partial t}$$

$$\therefore \boxed{\nabla^2\vec{H} - \frac{1}{c^2}\frac{\partial \vec{H}}{\partial t} = 0} \quad \text{— (B)}$$

(1)

[22-04-2020]

Eqn. (A) an (B) represents Wave Equations representing unattenuated wave travelling at a speed C (In a medium, the speed may be v) in terms of electromagnetic field E and H in free space. These are vector Equations of identical form, which means each of the six components of E and H separately satisfy the same scaler (u) eqn of the form

$$\nabla^2 u - \mu_0\epsilon_0 \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{— (i)}$$

This eqn resembles with general wave eqn $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$ — (ii)

Hence by Comparing (i) and (ii)

$$v = \frac{1}{\sqrt{\mu_0\epsilon_0}} \Rightarrow v = c = \frac{1}{\sqrt{4\pi\times10^{-7}\,\frac{wb}{A\text{-}m} \times 8.8542\times10^{-12}\,F/m}}$$

$$= 3\times10^{8}\ m/s$$

Now, we try to find the solution of eqn (A) & B or (i) & (ii) for Plane E.M.W. A plane wave is defined as a wave whose amplitude is the same at any point in a plane $\perp$r to a specific direction.

The soln. of these eqns in standard form may be written as

$$E_{(r,t)} = E_0\, e^{i(k\cdot r - \omega t)} \quad \text{—— (V)}$$

$$H_{(r,t)} = H_0\, e^{i(k\cdot r - \omega t)} \quad \text{—— (VI)}$$

$$u_{(r,t)} = u_0\, e^{i(k\cdot r - \omega t)} \quad \text{—— (VII)}$$

Where E_0, H_0 & u_0 are Complex amplitudes which ... and time, K = wave Propagation Vector $= \frac{2\pi}{\lambda}\hat{n}$

$K = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n}$; $\hat{n}$ - Unit Vector in the direction of Wave Propagation

$$\vec{\nabla}\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot E_0 e^{i(\vec{K}\cdot\vec{x} - \omega t)}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot \left\{(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})e^{i(K_x x + K_y y + K_z z - \omega t)}\right\}$$

$$= i\vec{K}\cdot\vec{E}$$

Similary we get $\vec{\nabla}\cdot\vec{H} = i\,\vec{K}\cdot\vec{H}$

From eqn (i) and (ii) we get

$i K\cdot E = 0$ —(viii) and $i K\cdot H = 0$ —(ix)

This means E.M. field vectors E and H are both perpendicular to the direction of propagation Vector K. This implies EMW are transverse in nature.

From eqn. (iii) $\nabla\times E = -\mu_0\frac{\partial H}{\partial t}$ and Eqn (iv) $\nabla\times H = \epsilon_0\frac{\partial E}{\partial t}$

Using eqn. (V) and (Vi), it Can be obtained

$i K\times E = -\mu_0(-i\omega H)$ or, $K\times E = \mu_0\omega H$ — (x)

$i K\times H = \epsilon_0(-i\omega E)$ or, $K\times H = -\epsilon_0\omega E$ — (xi)

from Eqn (x) and (xi), it is clear that H is $\perp_r$ to K and E both, and, Similarly E is $\perp_r$ to K and H both. So, E and H are mutually $\perp_r$ and also they both are $\perp_r$ to K (direction of Propagation of waves)

Vectors (E, H, K) form set of orthogonal vectors, which form a right handed Co-ordinate System in order.

From (x), $H = \frac{1}{\mu_0\omega}\cdot(K\times E) = \frac{K}{\mu_0\omega}(\hat{n}\times\vec{E})$

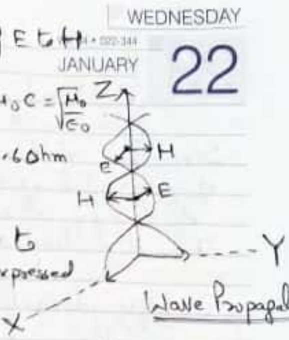
$$= \frac{1}{\mu_0 c}(n\times E)$$

(2)

The eqn ratio of magnitude of E to H is written as

$$Z_0 = \left|\frac{E}{H}\right| = \left|\frac{E_0}{H_0}\right| = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$= \sqrt{\frac{4\pi\times10^{-7}}{8.85\times10^{-12}}} = 376.6\,\text{ohm}$$



Wave Propagation

Z_0, measures the ratio of E (V/m) to H (amp-turns/m) and therefore expressed in terms of $\frac{\text{Volt}}{\text{Amp}}$ or ohm, and it is referred as "Wave Impedance". It is real as positive as it is real modi of two quantities. This means E and H are in phase, i.e both attain Max. or Min. Value at same time.

The Poynting Vector (Energy flow per unit area per unit time) for a plane EMW is given by

$$S = E\times H = E\times\frac{n\times E}{\mu_0 c} = \frac{1}{\mu_0 c}\vec{E}\times(\hat{n}\times\vec{E})$$

$$= \frac{1}{\mu_0 c}\left[(E\cdot E)c - \underbrace{(E\cdot n)}_{\text{Will be 0 as } \theta = \frac{\pi}{2}}E\right] = \frac{1}{\mu_0 c}E^2\hat{n}$$

$$= \frac{E^2}{Z_0}\hat{n}$$

For Plane EMW of angular frequency ω,

$\langle S\rangle$ = Average Value over one Cycle $= \frac{1}{Z_0}\langle E^2\rangle n$

$$= \frac{1}{Z_0}\left\langle\left\{E_0 e^{i(k\cdot r - \omega t)}\right\}^2_{real}\right\rangle\cdot n$$

$$= \frac{1}{Z_0}\frac{E_0^2}{2}n \quad \left[\text{Since } \langle\cos^2(\omega t - k\cdot r)\rangle = \frac{1}{2}\right]$$

$$\therefore \langle S\rangle = \frac{1}{Z_0} E_{rms}^2\, n \quad \text{— (xii)}$$

Direction of Poynting Vector is along the direction of propagation of EMW. This means that the flow of energy in a plane EMW in free space is along the direction of wave.

The ratio of Electrostatic and Magnetic energy densities

$$\frac{u_e}{u_m} = \frac{\frac{1}{2}\epsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\epsilon_0}{\mu_0}\frac{E^2}{H^2} = \frac{\epsilon_0}{\mu_0}\cdot\frac{\mu_0}{\epsilon_0} = 1$$

This means the ~~Electro~~ energy density is equal to magnetostatic energy density.

$$u = u_e + u_m = 2u_e = 2\times\frac{1}{2}\epsilon_0 E^2 = \epsilon_0 E^2$$

Time average of energy density

$$\langle u\rangle = \langle \epsilon_0 E^2\rangle = \epsilon_0 \langle (E_0 e^{i(K\cdot r-\omega t)})^2\rangle$$

$$= \epsilon_0 E_0^2 \langle \cos^2(\omega t - K\cdot r)\rangle = \frac{1}{2}\epsilon_0 E_0^2$$

$$= \epsilon_0 E_{rms}^2 \quad \text{— (xiii)}$$

From (xii) and (xiii)

$$\frac{\langle S\rangle}{\langle u\rangle} = \frac{1}{Z_0\epsilon_0} n = \frac{1}{\sqrt{\frac{\mu_0}{\epsilon_0}}\,\epsilon_0} n = \frac{n}{\sqrt{\mu_0\epsilon_0}} = c\cdot n$$

$$\langle S\rangle = \langle u\rangle\, c\, n$$

$\boxed{\text{Energy flux} = \text{Energy density} \times c}$ The energy density associated with an EMW in free space propagates with c with which the field vectors always propagate.

If we solve Maxwell's Eqn. for Non-conducting isotropic medium (dielectric) we get

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{K_m\mu_0 K_e\epsilon_0}} = \frac{c}{\sqrt{K_m K_e}}$$

where K_m = relative permeability and
K_e = relative permittivity of the medium.

$$v = \frac{c}{n} \text{ or } n = \text{R.I. (Refractive Index of the medium)} = \frac{c}{v}$$

$$\Rightarrow n = \sqrt{K_m K_e}$$

EMW in isotropic dielectric media are also transverse in nature with E and H being mutually $\perp$r and also they are $\perp$r to the direction of propagation vector K.

$$Z = \left|\frac{E}{H}\right| = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{K_m\mu_0}{K_e\epsilon_0}} = \text{Real quantity}$$

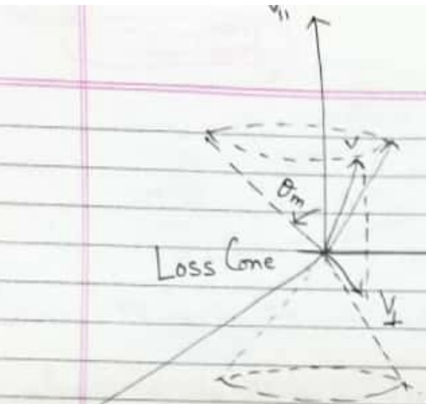
$$= \sqrt{\frac{K_m}{K_e}}\, Z_0 \quad \text{— (xiii)}$$

and $\langle S\rangle = \frac{n}{K_m}\cdot\langle S\rangle_{\text{freespace}}$ — (xiv)

Poynting Vector for EMW in isotropic dielectric is $\sqrt{\frac{K_e}{K_m}}$ or $\frac{n}{K_m}$ times Poynting vector in free space for same wave.

$$\text{Energy flux}\left(\langle s\rangle_{av}\right) = v \times \text{energy density}$$

The Energy flowing with velocity v (Phase velocity of EMW with which EM field vectors propagate), in the direction of propagation of wave, all the energy contained in a cylinder of unit cross-section and height equal to v would cross unit cross-section per sec.



Particles with smaller θ will mirror in the region of higher B. If θ is too small, B' exceeds B_m, and the particle does not mirror at all.

If θ is too small, B' exceeds B_m and replacing $B' \to B_m$ and $\sin^2\theta_m = \frac{B_0}{B_m} = \frac{1}{R_m}$ → (Mirror Ratio)

where θ_m = Smallest θ of a Confined Particle

This eqn. gives the boundary of a region in velocity space in the shape of a Cone, called "LOSS CONE". Particles lying within the Cone are not Confined; So, a mirror-confined Plasma can't be Isotropic. Loss-Cone is independent of q or m. Without Collision both types of particles are equally Confined. But under thermal Collisions, particles are lost when they change their pitch angle in a Collision and are scattered into the loss Cone. Due to smaller inertia, e^- are lost more easily due to their higher collision frequency.

The Concept of Magnetic Mirror was first proposed by Enrico Fermi as a mechanism for the acceleration of Cosmic rays. Protons gain energy with each bouncing between magnetic mirrors. This effect also is responsible for Confinement of particles in the Van-Allen belts. The strong B at pole and weak B at Equator forms natural mirror with rather large R_m.

The most promising device on this Concept is the "TOKMAK". In case of toroidal Pinch discharge (The inward Compression due to lateral repulsion b/w Concentric circled lines of forces, created by ionised system and its associated magnetic field is Known as PINCH EFFECT), the plasma is heated as well as remained Confined by the longitudinal Current flowing parallel to the major Circumference of the toroid. The strong toroidal magnetic field combines with the azimuthal B produced by the main current to form magnetic surfaces on which lie the individual helical lines of force. ~~Ano~~ The magnetic field produced due to Current flowing along Plasma ring, tries to expand the system and it is to be balanced by a field in Vertical direction (to counterbalance expansion).

The azimuthal Pressure $\left(\frac{B_\theta^2}{8\pi}\right)$ tends to increase major radius of Plasma. The vertical field B_v is directed along $\vec{J} \times \vec{B}$ (radially inward).

This Pinch Configuration has been used in TokmaK as well as Zeta apparatus. Zeta has only one weak toroidal field (B_z) while TokmaK has stronger field $B_z \simeq 10 B_\theta$. Ion densities of the order of $10^{13}/cc$ with temperature $T_e \simeq 1-2$ KeV and $T_i = 0.5$ KeV have been contained with Confinement time of $\simeq 10$ ms, which may increase with the toroid Current and ion density.

ADIABATIC EXPANSION;

In classical Mechanics Phase-Space Concept, $\oint p \cdot dq$, Action integral over a period is a Constant of Motion. p = generalised Momenta and q = generalized Co-ordinate and are taken as period functions. If a slow change (In Comparison to period of motion) is made in the system such that the motion is not quite periodic, the Constant of motion doesnot change then variables are called Adiabatic Invariant.

The variation is slow such that $\oint p\,dq$ remains well defined, but it no longer remains an integral over closed path. This concept of Adiabatic invariance is Very important to Plasma Physics. In Plasma, the magnetic moment $\mu = \frac{m v_\perp^2}{2B}$ remains invariant in Spatially and temporally magnetic fields. The period motion is Larmor Gyration.

If $p = m v_\perp r$ and $dq \to d\theta$ then action integral becomes

$$\oint p\,dq = \oint m v_\perp r \, d\theta = 2\pi r m v_\perp$$

$$= 2\pi \frac{m v_\perp^2}{\omega_c} = 4\pi \frac{m}{|q|} = \mu$$

where μ is a constant of motion as μ is invariant to all orders of ω/ω_c, ω = freq. characterisation of rate of change of B.

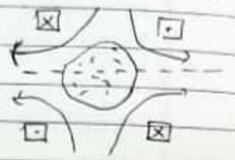
Adiabatic invariance of μ is violated where ω is not small compared to ω_c like in —

i) magnetic Pumping; If strength of B in a mirror Confinement is changing Sinusoidally, the invariance of μ is violated and plasma get heated due to Collision.

ii) Cyclotron heating; If B oscillates with ω_c, the induced E will rotate in Phase with some of the particles

The condition $\omega << \omega_c$ is violated, μ is not conserved and plasma may get heated.

(iii) Magnetic Cusps; If current in one of the coil in magnetic mirror system is reversed. It leads to spindle-cusp mirror extending over 360° in azimuth.



Since B vanishes at the centre of symmetry, $\omega_c = 0$ there and μ is not preserved. The local Larmor radius near the centre is larger than the device. Adiabatic invariant μ does not guarantee that particle outside a loss cone will stay outside after passing through the nonadiabatic region. This ensures trapping of particles until collision disturbs it.

If a particle is trapped between two magnetic mirrors, it bounces between them and has a periodic motion at the "bounce frequency". A constant of this type of motion is given by $\oint m v_{\parallel} \, ds$, where ds is an elementary path length along field line. Actually the guiding centre drifts across field lines and makes periodic motion ~~dot~~ to deviate from exact period. The constant of motion becomes an adiabatic invariant.

(6)

The longitudinal invariant J is defined for a half-cycle between two turning points as

$$\boxed{J = \int_a^b v_{\parallel} \, ds}$$

J remains invariant in a static, no-uniform field as well as for a slowly time-varying B field. Earth's magnetic field mirror-traps charged particle.

J-invariance violation occurs in transit-time magnetic pumping. If oscillating current is applied to the coils of a mirror system so that the mirrors alternatively approach and withdraw from each other near bounce frequency, then $v_{\parallel}$ increases and J is not conserved.

Slow drift of a guiding centre around the earth constitute a third type of periodic motion. The adiabatic invariant associated with this type of period motion leads to total magnetic flux Φ enclosed by the drift surface. As B varies, the particles stay on a surface will that the total no. of lines of force enclosed remain constant. Excitation of hydromagnetic waves in the Ionosphere is an example of the violation of Φ.