Magnetohydrodynamics

Course: MPHYEC-01 Plasma Physics
(M.Sc. Sem-IV)

Dr. Sanjay Kumar

Assistant Professor
Department of Physics
Patna University
Contact Details: Email- sainisanjay35@gmail.com;
Contact No.- 9413674416

Unit-02

Magnetohydrodynamics

Magnetohydrodynamics Equations

From Unit-I, we know that plasma can formally be defined as a collection of charged and neutral particles which obeys the condition of quasi-neutrality and exhibits collective behavior in presence of self-consistent electromagnetic field. Note that due to the self-consistent nature of the involved electromagnetic force, in general, a direct understanding of dynamics of these charged particles is difficult. Then, a reasonable simplification is achieved by using a fluid (continuum) approximation where an individual particle looses its identity. Under this approximation, it is assumed that plasma is made of infinitesimal fluid elements (sometimes also known as fluid parcels) that have dimension which is much smaller than the size of whole plasma system (in laboratory plasma, you can think of the size of the container in which plasma is filled), however the dimension is much greater than the inter-molecular distances so that the a fluid element have a large number of particles.

Under this approximation, for sufficiently large time and length scales the plasma can be treated as a magnetized fluid or magnetofluid. The dynamics of the fluid is then described by magnetohydrodynamics (MHD) equations, which are generically similar to the Navier-Stokes equations complemented by the Maxwell's equations in their non-relativistic limit. The MHD equations (in SI units) are

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -\nabla p + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}, \quad \text{----- (1)}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \quad \text{---- (2)}$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J}, \quad \text{---- (3)}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{---- (4)}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}, \quad \text{---- (5)}$$

$$\nabla\cdot\mathbf{B} = 0, \quad \text{---- (6)}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0, \quad \text{---- (7)}$$

in standard notations (see any standard plasma books), where electrical resistivity η and dynamic viscosity μ are assumed to be constant. To describe, $\boldsymbol{v}$ represents plasma velocity. ρ is mass density, denoting plasma mass per unit volume. $\boldsymbol{B}$ and $\boldsymbol{E}$ are magnetic and electric fields. p represents plasma pressure (similar to the one used in thermodynamics). Equation (1) is known as the momentum transport equation which is basically Newton's second law for the fluid parcels of plasma and describes their motion. Equation (2) is mass continuity equation which represents the law of mass conservation. Equation (3) is basically Ohm's law for moving conductor (as plasma can be treated as a conductor in motion). Equations (4) and (5) are 3rd and 4th Maxwell's equations and represent Faraday's law and Ampere's law. Note that the displacement current is neglected in Ampere's law (equation (5)). This is due to the assumption that plasma velocity are non-relativistic (i.e. v << c). Important to note here is that the fluid description of plasma is achieved by taking moments of the Vlasov equation. In this process the nth moment, for any arbitrary n, always contains a term involving (n+1)th moment. This forbids the corresponding continuum equations to get closed and necessitates the requirement of an ad hoc closure. Typically an equation of state, usually utilized in thermodynamics, is assumed to formally close the MHD equations. The equation (7) is one such closure relating thermodynamic pressure and mass density.

Importantly, the MHD equations are always non-linear. To appreciate the non-linearity, we substitute electric field **E** from the Ohm's law (3) to equation (4) and get,

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \eta \boldsymbol{J} + \nabla \times \boldsymbol{v} \times \boldsymbol{B}$$

now plug the current density **J** from equation (5) into above equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\eta}{\mu_0} \nabla \times (\nabla \times \boldsymbol{B}) + \nabla \times \boldsymbol{v} \times \boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \boldsymbol{v} \times \boldsymbol{B} + \lambda \nabla^2 \boldsymbol{B} \qquad \text{--------- (8)}$$

where $\lambda = \eta/\mu_0$ is known as magnetic diffusivity. Equation (8) basically represents the combined form of equations (3), (4) and (5). This equation is known as "*induction equation*". *Therefore, equations (1), (2), (7) and (8) form complete set of MHD equations.*

Important to note is that due to the presence of first term in left-hand-side (LHS) of induction equation, velocity (**v**) and magnetic field (**B**) become coupled and non-linear through the momentum trasport equation (1). That's why the first term of induction equation is sometimes known as "non-linear" term while the second term is known as "diffusion" term.

To further scrutinize the importance of the induction equation in magnetofluid evolution, we first look at this equation carefully before discussing other aspects of the MHD model. If B is the typical magnetic field, v the typical velocity and L the typical length scale of a plasma system, then the non-linear term ($\nabla \times \mathbf{v} \times \mathbf{B}$) of induction equation is of order vB/L and the diffusion term ($\lambda\nabla^2\mathbf{B}$) is of order $\lambda B/L^2$. The ratio of non-linearity to diffusion yields magnetic Reynolds number

$$R_M = |\nabla \times \mathbf{v} \times \mathbf{B}| \,/\, |\lambda\nabla^2\mathbf{B}|$$

$$R_M = (vB/L) \,/\, (\lambda B/L^2)$$

$$R_M = vL/\lambda \qquad \text{----------} \ (9)$$

The value of R_M determines the effectiveness of the non-linearity over the diffusion in an evolving magnetofluid. For instance, if $R_M >> 1$, the non-linearity dominates over the diffusion. Whereas for $R_M << 1$, the magnetofluid becomes predominantly diffusive. Since R_M is directly proportional to the size L of the system, it turns out to be much larger for astrophysical plasmas compared to laboratory plasmas.

Typically, for a hydrogen plasma of temperature 10^4 K, magnetic diffusivity λ is approximately equal to 10^7 m^2/s. Taking this value of λ, we now estimate R_M for a laboratory system and for an astrophysical system. If we take L = 10^2 cm and v= 10 cm/s for a typical laboratory system, then we find from (9) that $R_M = 10^{-4}$. Now let us consider granules or convection cells on the solar surface, which are very small objects by astrophysical standards. Using the typical values L=10^8 cm and v= 10^5 cm /s, we find $R_M = 10^6$. We therefore conclude that the magnetic Reynolds number is generally small (<< 1) for laboratory systems and very large (>> 1) for astrophysical systems.

Ideal Magnetohydrodynamics

In the limit of $R_M >> 1$ (which is the case for most of the astrophysical and space plasma systems), the magnitude of diffusion term is much smaller than the non-linear term. Therefore, in this limit of high Reynolds number, the induction equation reduces as:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \boldsymbol{v} \times \boldsymbol{B} \qquad \text{----------} \ (10)$$

With the difusion term being neglected in the induction equation, the MHD equations are often known as **ideal MHD** equations. It is worthy to mention here that ideal MHD imposes certain constraints on the evolution of plasma. To understanding the constraints, first, we discuss the concept of magnetic field lines (MFLs) which very important in plasma.

Relevantly, the magnetic field **B** can be represented by magnetic field lines. A magnetic field line is a space-curve which is everywhere tangential to a given magnetic field **B** = B_x $\hat{\mathbf{e}}_x$ + B_y $\hat{\mathbf{e}}_y$ + B_z $\hat{\mathbf{e}}_z$ and is described by the ordinary differential equations (ODEs):

$$\frac{dx}{ds} = \frac{B_x}{|\,\mathbf{B}\,|},$$
$$\frac{dy}{ds} = \frac{B_y}{|\,\mathbf{B}\,|},$$
$$\frac{dz}{ds} = \frac{B_z}{|\,\mathbf{B}\,|},$$

in Cartesian coordinates (x, y, z), where ds is the invariant length and | B | is the magnitude of B. Field lines are obtained by integrating these differential equations. The topology of the magnetic field **B** is determined by the linkage and knottdness of magnetic field lines which do not change under continuous deformations.

Now, under ideal MHD, and the plasma satisfies the Alfvén's theorem of flux-freezing or frozen-in-condition, which ascertains the field lines to be tied with fluid parcels as the whole system evolves in time. As a consequence, the magnetic flux across an arbitrary fluid surface, physically identified by the material elements lying on it, remains conserved in time. Importantly, the otherwise abstract magnetic field lines attain physicality as the flux-freezing allows them to be identified with fluid parcels, which are real.

A proof of the flux-freezing is as follows. Let us consider an arbitrary fluid surface S enclosed by a curve C, moving with fluid velocity v. The magnetic flux passing through the surface S is given by,

$$\Phi = \int_S \mathbf{B} \cdot \, d\mathbf{S}.$$

The rate of change of Φ is

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_C \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l},$$

where **dl** is the line element of C and **v** × **dl** is the area swept out by **dl** per unit time. Hence, the total rate of change of Φ includes two terms; the first term is due to the change in **B** over the surface S and the second term is due to the variation in area spanned by the S as the boundary C moves in space. Employing Stokes' theorem in the above equation we get,

$$\frac{d\phi}{dt} = \int_S \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right) \cdot d\mathbf{S}.$$

Now, using equation (10) in the above equation, it is straightforward to show that

$$\frac{d\phi}{dt} = 0,$$

stating that the φ is conserved across the fluid surface S. As a consequence, depending on the topology of magnetic field, few fluid surfaces can be identified such that the magnetic field lines are entirely contained on the fluid surfaces which, in literature are termed as magnetic flux surfaces (MFSs). It is then imperative that the magnetic flux passing through the fluid surfaces is zero. Under the condition of flux-freezing, the fluid surfaces retain this zero flux during the evolution which ensures that the field lines always remain tangential to the fluid surfaces. This infers that the magnetic field lines are tied to the fluid parcels physically identifying the fluid surfaces and hence magnetic flux surfaces can be associated with the fluid surfaces. This association is maintained throughout the evolution under the flux-freezing.

Another important impact of flux-freezing on the dynamics is that if two fluid parcels are connected by a field line at time t = 0, they will always remain connected by the same field line. The preservation of such connectivities warrants the invariance of topological properties of the field lines; e.g., if two field lines are linked n times at t = 0, then they must maintain the same number of linkages at all times.

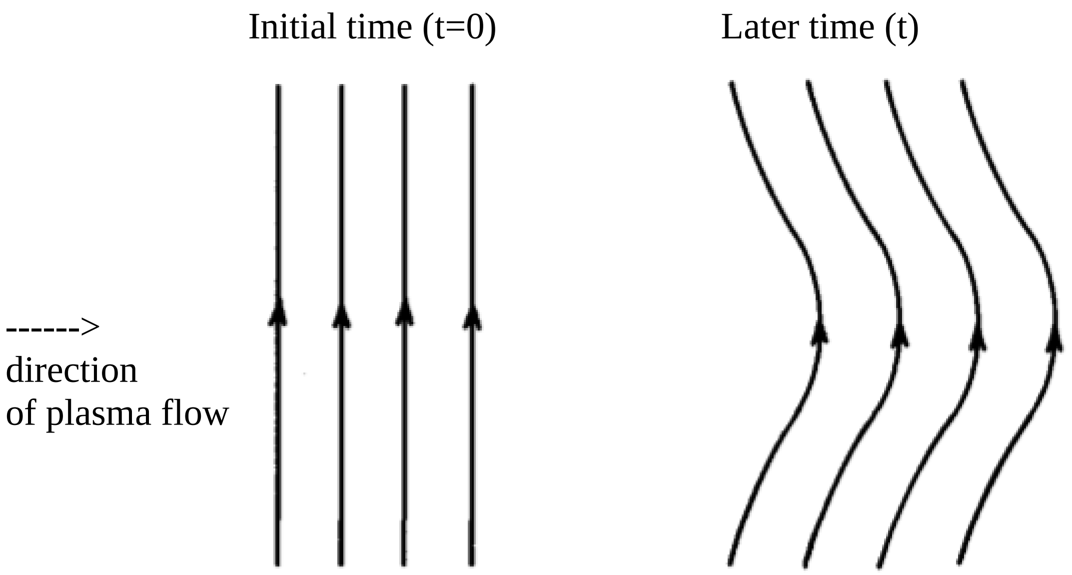

Figure- 1

The figure 1 (above) shows a pictorial demonstration of the flux-freezing or frozen-in condition. From the figure, the field lines which are initially straight get bend due to the motion of plasma.

Further illustration of the flux-freezing condition is shown in figure 2. Let A and B be two fluid elements which lie on the same magnetic field line as shown in figure 2. We consider a thin cylindrical surface around this field line. It is evident that the magnetic flux across this cylindrical surface is zero. After some time, the fluid elements A and B take up positions C and D , whereas the fluid elements which made the previous cylindrical surface now make up a different cylindrical surface as shown in second panel of figure 2. According to the flux-freezing, the magnetic flux through this new cylindrical surface should be zero, and this is possible only if the field line still passes along the axis of the cylindrical surface. This means that C and D still lie on the same magnetic field line. In other words, if two fluid elements are connected by a field line, they will always remain connected by a field line in the limit of ideal MHD.

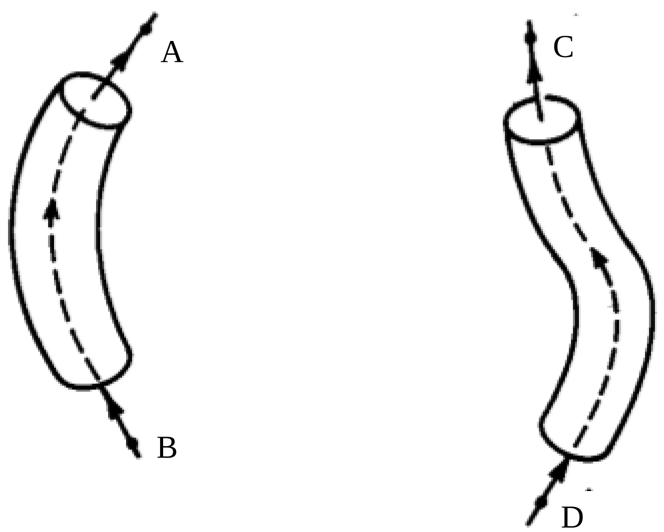

Figure 2

Reference: Book "Solar Magnetohydrodynamics" by Eric Priest.

Thanks for the attention!