Linear Med- $\nabla \cdot D = \rho$

MPHYC – 6

For free space

$\nabla \cdot B = 0$, $B = \mu_0 H \Rightarrow \nabla$

$\nabla \cdot E = 0$

$\nabla \times E = -\frac{\partial B}{\partial t} =$

$\nabla \cdot B = 0$

$\nabla \times E = -\frac{\partial B}{\partial t}$

Unit – I

$\nabla \times H = J + \frac{\partial D}{\partial t}$

$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$

E. M. Energy

Electrostatic P.E. $= \frac{1}{2}\int_V E \cdot D \, dv$

Energy stored in a magnetic field $= \frac{1}{2}\int_V H \cdot B \, dv$

$$\underbrace{\int_V \operatorname{div}(E \times H) \, dV}_{\oint_S (E \times H) \cdot ds} = -\int_V \left\{ \frac{\partial}{\partial t}(E \cdot D + H \cdot B) \right\} dv - \int_V J \cdot E \, dv$$

$$\therefore \underbrace{-\int_V J \cdot E \, dV}_{\substack{\text{Rate of Energy transferred in to the}\\ \text{EM field through the motion of free charge}\\ \text{in volume 'V'}}} = \underbrace{\frac{d}{dt}\int_V \frac{1}{2}(E \cdot D + H \cdot B) \, dV}_{\substack{\text{Rate of E.M. Energy}\\ \text{Stored in Volume V}}} - \oint_S \cancel{E \cdot E \, dv} \, (E \times H) \cdot ds$$

$$\nabla \cdot (E \times H) = -E \cdot (\nabla \times H) + H (\nabla \times E)$$

$$= -\mu_0 \left[E \epsilon_0 \frac{\partial E}{\partial t} + H \cdot \frac{\partial H}{\partial t} \right]$$

$$= -\mu_0 \frac{\partial}{\partial t}\left[\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right] \rightarrow$$ Rate of change of Electric & Magnetic Energy associated with EM field vectors.

(2)

Interpretation of $\oint (E \times H) \cdot ds$

The Surface integral in this term involves only electric and magnetic fields, it is ~~possible~~ may be interpreted as the rate of energy flow across the surface. So, $(E \times H)$ itself represents the energy flow per unit time per unit area.

Surface integral of $(E \times H)$ over a closed surface represents the amount of EM Energy crossing the closed surface per second.

Let us Consider a finite volume surface 'S' such that within a given interval of time, none of the charged particles cross the surace. For Conservation of charge

$$-\oint (E \times H) \cdot ds = \underbrace{\frac{\partial U}{\partial t} + \frac{\partial W}{\partial t}}$$

Energy flowing out of the volume bounded by the surface per sec.

Time-Rate of change of field and of particles Contained Within the Volume

Since no particle is ~~flowing~~ Crossing the Surface, $(E \times H)$ is interpreted as the amount of field energy passing through unit area of the Surface in unit time, which is normal to the direction of energy flow.

$$S = E \times H$$

(3)

$$\therefore -J \cdot E = \frac{\partial U}{\partial t} + \nabla \cdot S$$

The Physical meaning of this equation is that the time rate of change of E.M Energy with a Certain volume plus time rate of the energy flowing out through the boundary surface is equal to the Power transferred into the E.M field.

Poynting Vector $(S) = E \times H$

$\boxed{\text{By Maxwell}}$ Poynting vector introduces the Conception of energy of electromagnetic field as residing in the medium itself just like elastic potential energy of solids.

The electrical energy representing the P.E. of strain of the medium while magnetic energy K.E of motion, can't be compartmentalised.

Energy is regarded as localised in space and as travelling in manner given by Poynting Vector.

In EMW, there is certain Energy per unit Volume, proportional to the square of the amplitude (E or H). Poynting Vector measures the rate of flow or intensity of ~~flow~~ the wave. In plane EMW, E & H are at ⊥ to each other and at right angles to direction of flow. Therefore E×H must be along the direction of flow. In more Complicated waves, Poynding Vector points along the direction of flow of Radiation

$[H] = \frac{[A][L][L]}{[L]^3} = [AL^{-1}]$

$[S] = [E][H] = [MT^{-3}]$

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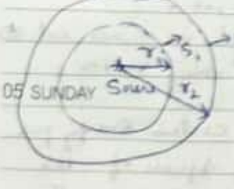
In Case of time varying fields $S = E \times H$ gives the instantaneous value of the Poynting Vector.

S has magnitude $EH \sin 90^\circ = EH$ and points along ~~field~~ wave propagation. Its dimension is $\frac{\text{Energy}}{\text{Area} \times \text{Time}}$ and units $\frac{J}{m^2 \cdot s} = \frac{\text{Watt}}{m^2}$ *

Characteristics of Poynting Vector

① $S \propto \frac{1}{r^2}$

Poynting Vector at any arbitrary point in the field varies inversely as the square of the distance from Point Source of Radiation



$S_1 \times 4\pi r_1^2 = S_2 \times 4\pi r_2^2$

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② The definition of Poynting Vector is not mandatory

Since this vector has been introduced as divergence, the curl of any arbitrary vector can be added

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to it without altering the Physical fact of the case.

So, $S = E \times H + G$

where $G = \text{Curl } M$

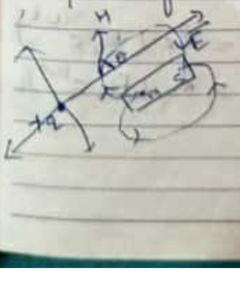
M = Any arbitrary Vector

The additional term has no physical Consequences and definition of S ~~can be~~ remain unaltered by such additional term.

③ If Poynting Vector is zero, then no EM Energy can flow across a closed surface but if no net field energy is ~~zero~~ flowing across a closed surface, the Poynting vector may or may not be zero

For Ex.

Field due to point charge in the presence of a magnet at rest 'or' charged capacitor placed b/w poles of a permanent magnet if E is not parallel to H then Poynting vector



$|\vec{S}| = |\vec{E} \times \vec{H}| = EH \sin\theta \neq 0$

For steady field $S \neq 0 \Rightarrow \oint S \cdot ds$ but there is no flow of energy across the surface

$\oint S \cdot ds = 0$

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(6)

Conversely,
If the flow of energy across closed surface is zero as

$$\oint S \cdot ds = \int_\tau \nabla \cdot S \, d\tau = \int_\tau \nabla \cdot (E \times H) \, d\tau$$

$$= \int_\tau (H \cdot \text{Curl} E - E \cdot \text{Curl} H) \, d\tau = 0$$

As E and H are Constant for steady fields.

It is so because S does not determine rate of flow through small elements ds but implies that only the flux of S across a closed surface is significant.

(4) Let us Consider Time - Varying field, i.e. E and H be given by real parts of Complex exponentials of the form,

$$E = E_0(r)\, e^{i\omega t} = Re\,(E_0 e^{i\omega t}) = Re\,[(E_r + j E_{im})(\cos\omega t + j \sin\omega t)] = E_r \cos\omega t - E_{im} \ldots$$

$$H = H_0(r)\, e^{i\omega t} = Re\,(H_0 e^{i\omega t}) = Re\,[(H_r + j H_{im})(\cos\omega t + j \sin\omega t)] = H_r \cos\omega t - H_{im} \sin \ldots$$

(7)

The Poynting Vector is

$$S = E \times H$$

$$= (E_r \times H_r) \cos^2\omega t + (E_{im} \times H_{im}) \sin^2\omega t$$

$$- [(E_r \times H_{im}) + (E_{im} \times H_r)] \sin\omega t \cos\omega t$$

This expression has two parts — (i) The time average of first two terms are non-zero, since average of $\sin^2\omega t$ and $\cos^2\omega t$ is $\frac{1}{2}$, and time average of last term is zero. Therefore time average of Poynting vector over a complete cycle is

$$\langle S \rangle = \langle E \times H \rangle = \frac{1}{2}[(E_r \times H_r) + (E_{im} \times H_{im})]$$

— (x) $\Rightarrow$

Under Complex Conjugate notation,

$$(E \times H^*) = (E_0 e^{i\omega t}) \times (H_0^* e^{-i\omega t})$$

$$= E_0 \times H_0^*$$

$$= ((E_r + jE_{im}) \times (H_r - jH_{im}))$$

$$= (E_r \times H_r + E_{im} \times H_{im})$$

$$+ j(E_{im} \times H_r - E_r \times H_{im}) \qquad \text{— (Y)}$$

Comparision of (X) and (Y), shows that, except factor 1/2, the ~~main~~ real part of Eqn. (Y) and (X) are same.

$$\langle S \rangle = \langle E \times H \rangle = \frac{1}{2} R_e (E \times H^*)$$

The instantaneous value of field is given by

$$\langle S \rangle = \frac{1}{T} \int_0^{2\pi} (E \times H)\, dt$$

$$= \frac{1}{2} (E_0 \times H_0) = \frac{E_0}{\sqrt{2}} \times \frac{H_0}{\sqrt{2}} = E_{av} \times H_{av}$$

The importance of Poynting Vector can be visualised, in the interpretation of various optical phenomena like reflection, refraction, dispersion, scattering, diffraction etc.

Momentum in E.M. Field ;

The E.M. field possesses energy must posses mass. The Energy per unit volume of E.M. field is given by.

$$W = \frac{1}{2}\left(\epsilon E^2 + \mu H^2\right)$$

The ~~mass~~ inherent / associated mass is given by,

$m = \frac{W}{c}$. Since, the momentum 'p' is given by $p = mc$ per unit Volume.

Let us Consider an arbitary Region of Volume τ surrounded by a Surface 'S' and Containing charges and currents. The field in the region of Consideration Will be governed by Maxwell's Eqn.

~~Div~~ $\nabla\cdot D = \rho$ $\qquad \nabla\cdot B = 0$

$\nabla\times E = -\frac{\partial B}{\partial t}$ $\qquad \nabla\times H = J + \frac{\partial D}{\partial t}$

The total force on the matter is given by Lorentz force equation, as

$$F = \int_\tau (\rho E + J\times B)\, d\tau$$

$$= \int_\tau \left\{E(\nabla\cdot D) + \left[\nabla\times H - \frac{\partial D}{\partial t}\right]\times B\right\} d\tau$$

Consider

$\nabla\cdot B = 0 \Rightarrow H(\nabla\cdot B) = 0$ Multiply by

$\nabla\times E = -\frac{\partial B}{\partial t} \Rightarrow -D\times\left\{(\nabla\times E) + \frac{\partial B}{\partial t}\right\} = 0$

$\nabla\times E + \frac{\partial B}{\partial t} = 0$

Multiplying by $-D$ (vector product of $-D$ with Eqn)

Adding these two Eqns yield

$$H(\nabla\cdot B) - D\times\left\{(\nabla\times E) + \frac{\partial B}{\partial t}\right\} = 0$$

Lorentz force may be written as

$$\text{So, } F = \int_\tau \left\{E(\nabla\cdot D) + \left(\nabla\times H - \frac{\partial D}{\partial t}\right)\times B - D\times\left((\nabla\times E) + \frac{\partial B}{\partial t}\right) + H\,\nabla\cdot B\right\} d\tau$$

$$= \int_\tau \left[\left\{E(\nabla\cdot D) - D\times(\nabla\times E)\right\} + \left\{H\,\nabla\cdot B - B\times(\nabla\times H)\right\} - \frac{\partial}{\partial t}(D\times B)\right] d\tau$$

Since $D = \epsilon_0 E$, $B = \mu_0 H$

$$= \int_\tau \left[\epsilon_0\left\{E(\nabla\cdot E) - E\times(\nabla\times E)\right\} + \mu_0\left\{H(\nabla\cdot H) - H\times(\nabla\times H)\right\} - \epsilon_0\mu_0\frac{\partial}{\partial t}(E\times H)\right] d\tau$$

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Let us Consider x - Component of R.H.S only

$$\left. E(\nabla\cdot E) - E\times(\nabla\times E)\right|_x = E_x\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$-\left\{E_y\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) - E_z\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\right\} = \underbrace{\frac{\partial}{\partial x}(E_x^2) - E_x\frac{\partial E_x}{\partial x} - E_y\frac{\partial E_y}{\partial x} - E_z\frac{\partial E_z}{\partial x}}$$

$$+\frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z)$$

$$= \frac{\partial}{\partial x}\left(E_x^2 - \frac{1}{2}E^2\right) + \frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z)$$

$$\nabla\cdot E = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}E_x + \hat{j}E_y + \hat{k}E_z\right)$$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla\times E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\bar{E}\times(\bar{\nabla}\times\bar{E}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ E_x & E_y & E_z \\ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{vmatrix}$$

$$= \hat{i}\underbrace{\left\{E_y\cdot\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) - E_z\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\right\}}_{x\text{-Component}}$$

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